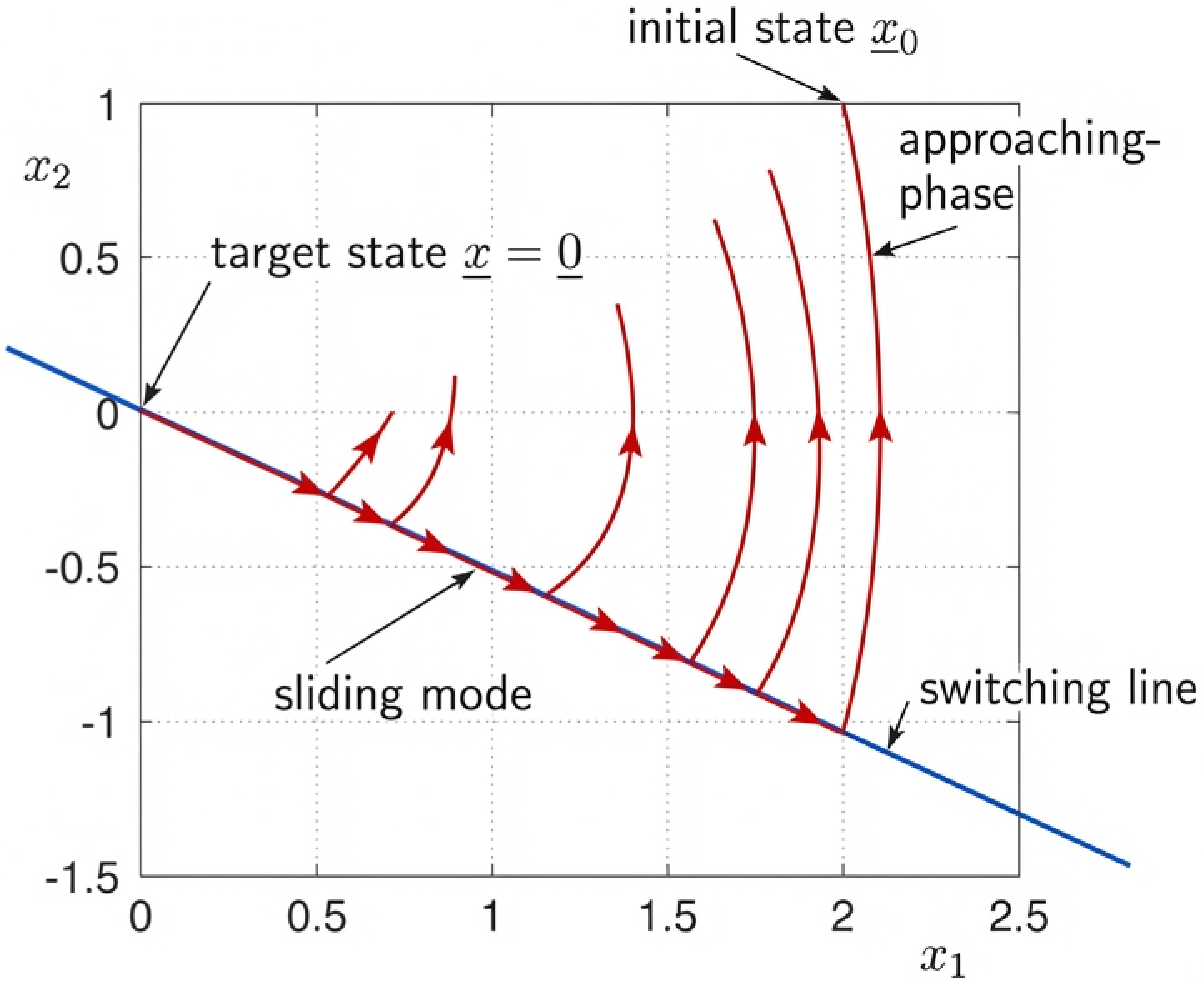
SLIDING MODE CONTROL

Dynamical Systems – Tutorial 10 Summary



Based on Lecture Notes: DS_10_SlidingMode.pdf

THE PRINCIPLE OF OF SLIDING MODE

System Definition

$$\dot{x} = f(x, u) + d(x, t)$$

$d(x, t)$ represents uncertainties/disturbances.

The Goal

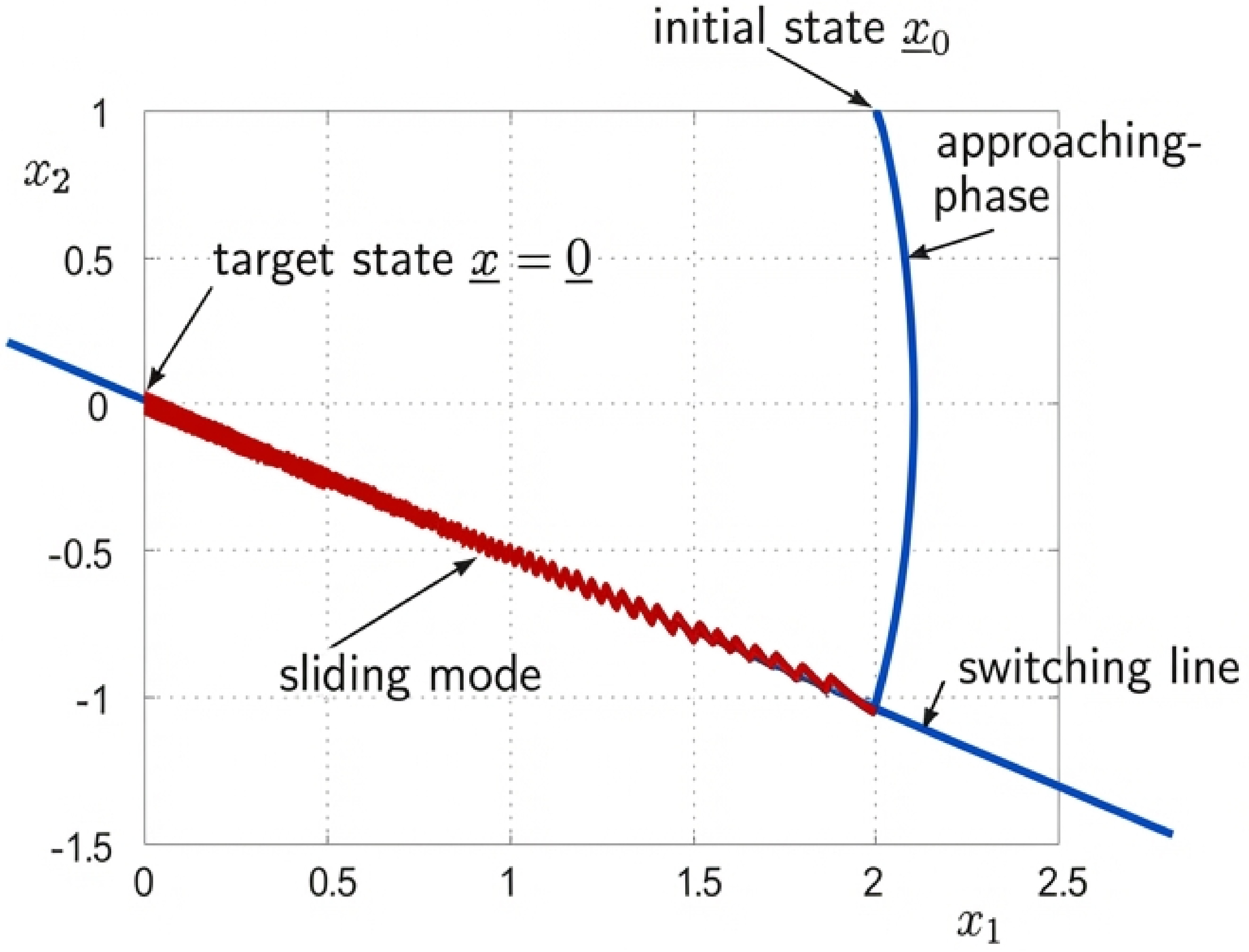
1. Reach manifold $S = \{x \in \mathbb{R}^n \mid s(x) = 0\}$ in finite time.
2. Remain on manifold S $(s(x) = 0)$.

The Controller

Discontinuous switching control:

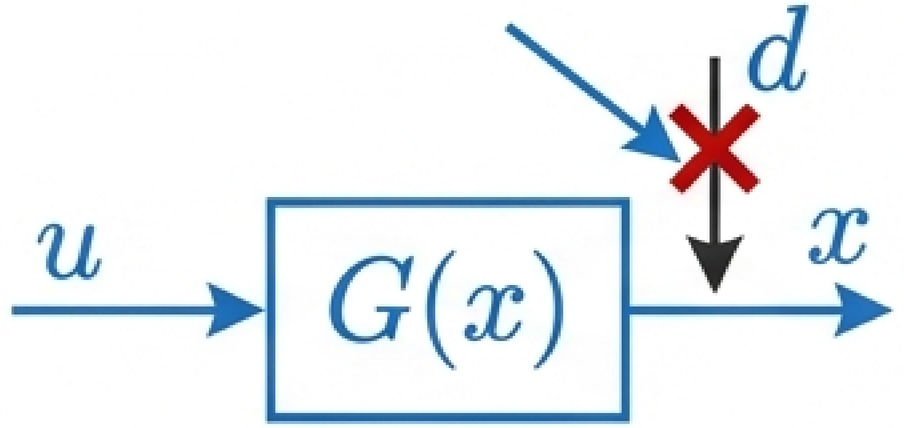
$$u(x) = u^+(x) \quad \text{for } s(x) > 0$$
$$u(x) = u^-(x) \quad \text{for } s(x) < 0$$

Constraint: $u^+(x) \neq u^-(x)$



CHARACTERISTICS & ROBUSTNESS

Invariance



System is highly robust against parameter fluctuations and external disturbances.

Matched Uncertainty: Disturbances must act in the control channel (span of $G(x)$).

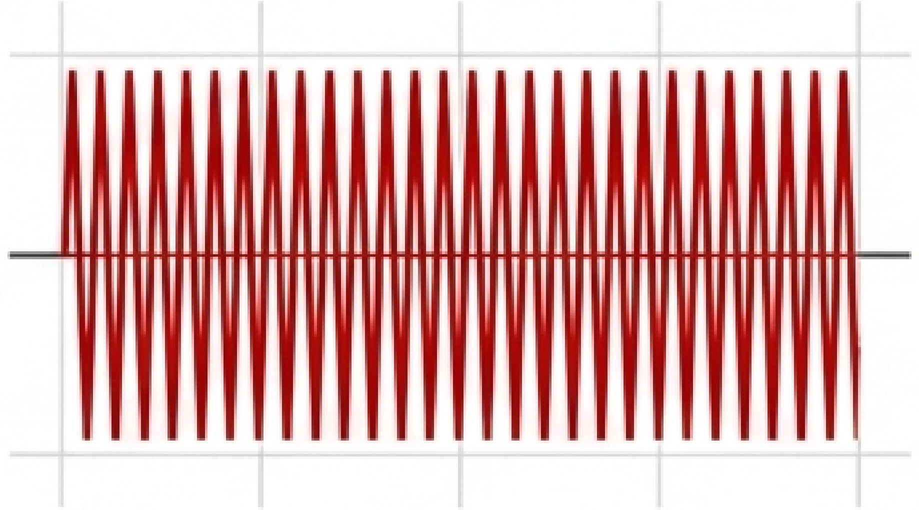
Reduced Dynamics

$$n \to n - m$$

In sliding mode, dynamics are reduced by order m.

Behavior is determined strictly by the manifold S, independent of original system parameters.

Chattering



Ideal sliding requires infinite switching frequency.

Reality: Finite frequency causes high-frequency oscillations (chattering).

Impact: High wear on actuators and physical stress.

EXISTENCE OF SLIDING MODE

Lyapunov Stability Analysis

Lyapunov Candidate: $V = 0.5 \cdot s^2$

Time Derivative: $\dot{V} = s \cdot \dot{s}$

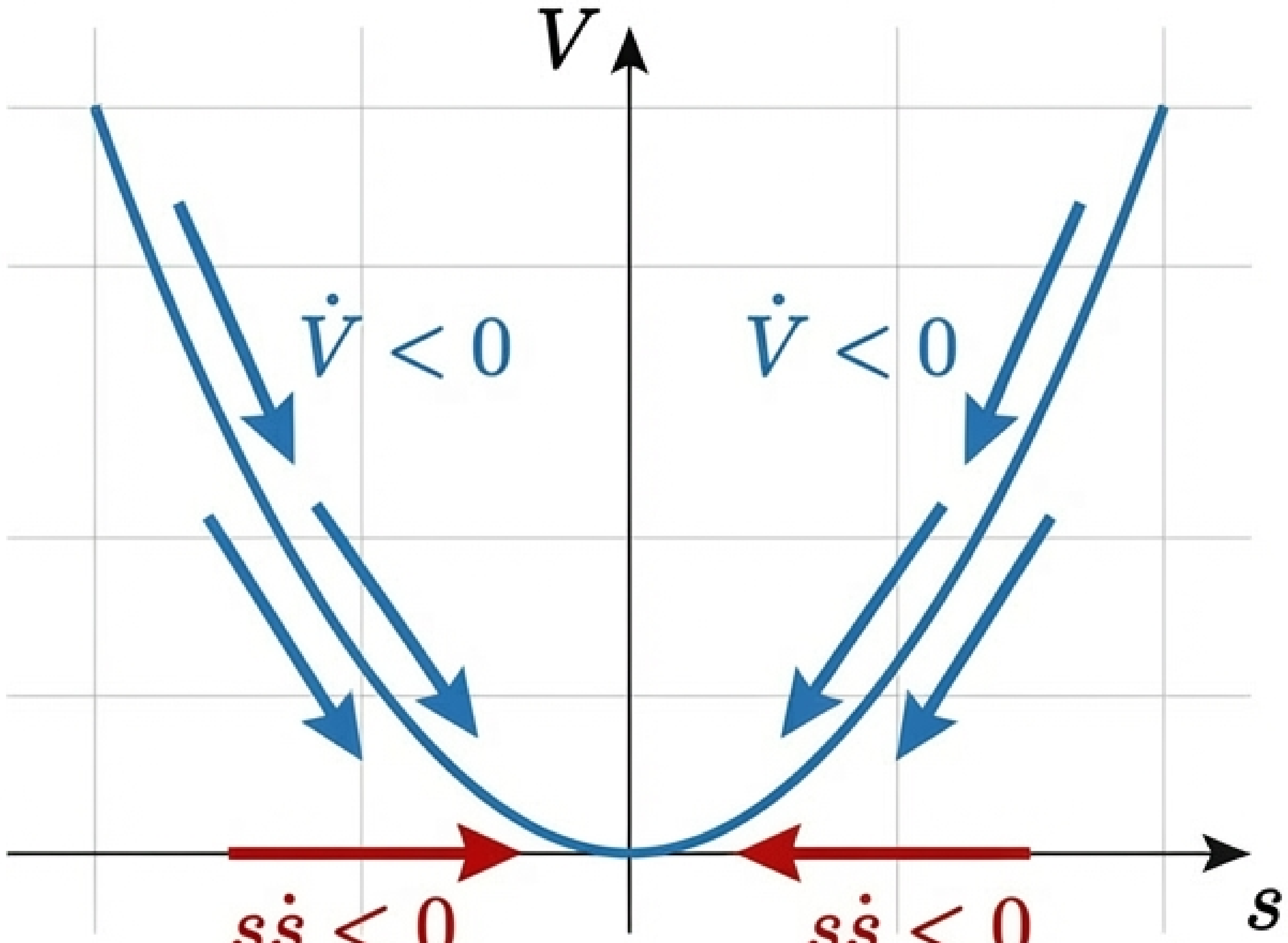
$$\boxed{s \cdot \dot{s} \leq -\eta \cdot |s|, \text{ for some } \eta > 0}$$

Reaching Time Bound: $t_s \leq \dfrac{||s(0)||}{\eta}$

Alternative Local Condition:

$$\lim_{s \to 0^-} \dot{s} > 0$$

$$\lim_{s \to 0^+} \dot{s} < 0$$



ROBUST DESIGN EXAMPLE: SETUP

SYSTEM & BOUNDS

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = f(x) + g(x)u$$

Uncertainties with Known Bounds:

$$|f(x)| \leq f_0(x)$$
$$g(x) \geq g_0 > 0$$

STEP 1 – SLIDING VARIABLE

Design Manifold:

$$s(x) = a \cdot x_1 + x_2, \quad \text{where } a > 0$$

STEP 2 – STABILITY ANALYSIS

Analyze $\dot{V} = s \cdot \dot{s}$:

$$\begin{aligned} s \cdot \dot{s} &= s(a \cdot \dot{x}_1 + \dot{x}_2) \\ &= s(a \cdot x_2 + f(x) + g(x)u) \end{aligned}$$

Inequality (Worst Case):

$$\begin{aligned} s \cdot \dot{s} \leq |s| \cdot (a \cdot |x_2| + f_0(x)) \\ + s \cdot g(x) \cdot u \end{aligned}$$

ROBUST DESIGN EXAMPLE: CONTROLLER

Step 3: Controller Design

Proposed Control Law:

$$u(x) = -\beta(x) * \mathrm{sign}(s)$$

Substitution & Analysis

Substitute $u(x)$ into the stability inequality:

$$s * \dot{s} \leq |s| * \left(a|x_2| + f_0(x) - g(x) * \beta(x)\right)$$

Enforcing the Condition

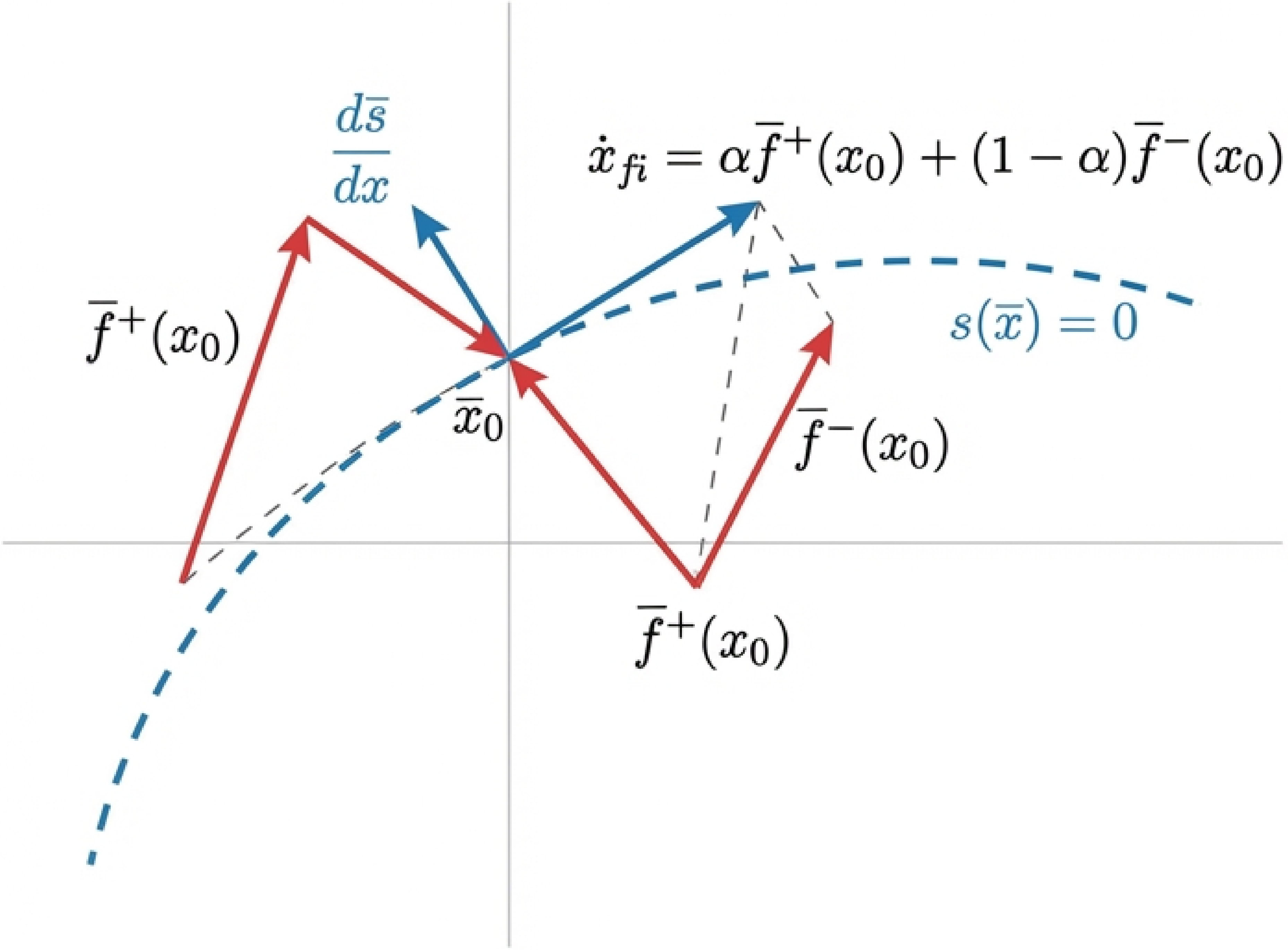
Requirement: $s * \dot{s} \leq -\eta * |s|$

Using the lower bound $-g(x) \leq -g_0$:

$$\boxed{\beta(x) \geq \frac{1}{g_0}\left(a \cdot |x_2| + f_0(x) + \eta\right)}$$

Global existence of sliding mode is guaranteed for any $f(x), g(x)$ within bounds.

DYNAMICS OF THE IDEAL SLIDING MODE



Problem:
System is not Lipschitz continuous at $s(\bar{x}) = 0$.

Ideal Condition:

1. $s(\bar{x}) = 0$
2. $\dot{s}(\bar{x}) = 0$

Two Analytical Methods:

1. Filippov's Method (Geometric)
2. Equivalent Control Method (Algebraic)

METHOD 1: FILIPPOV'S METHOD

Concept:

Solution lies in the convex hull of vector fields f^+ and f^-.

Dynamics Equation:

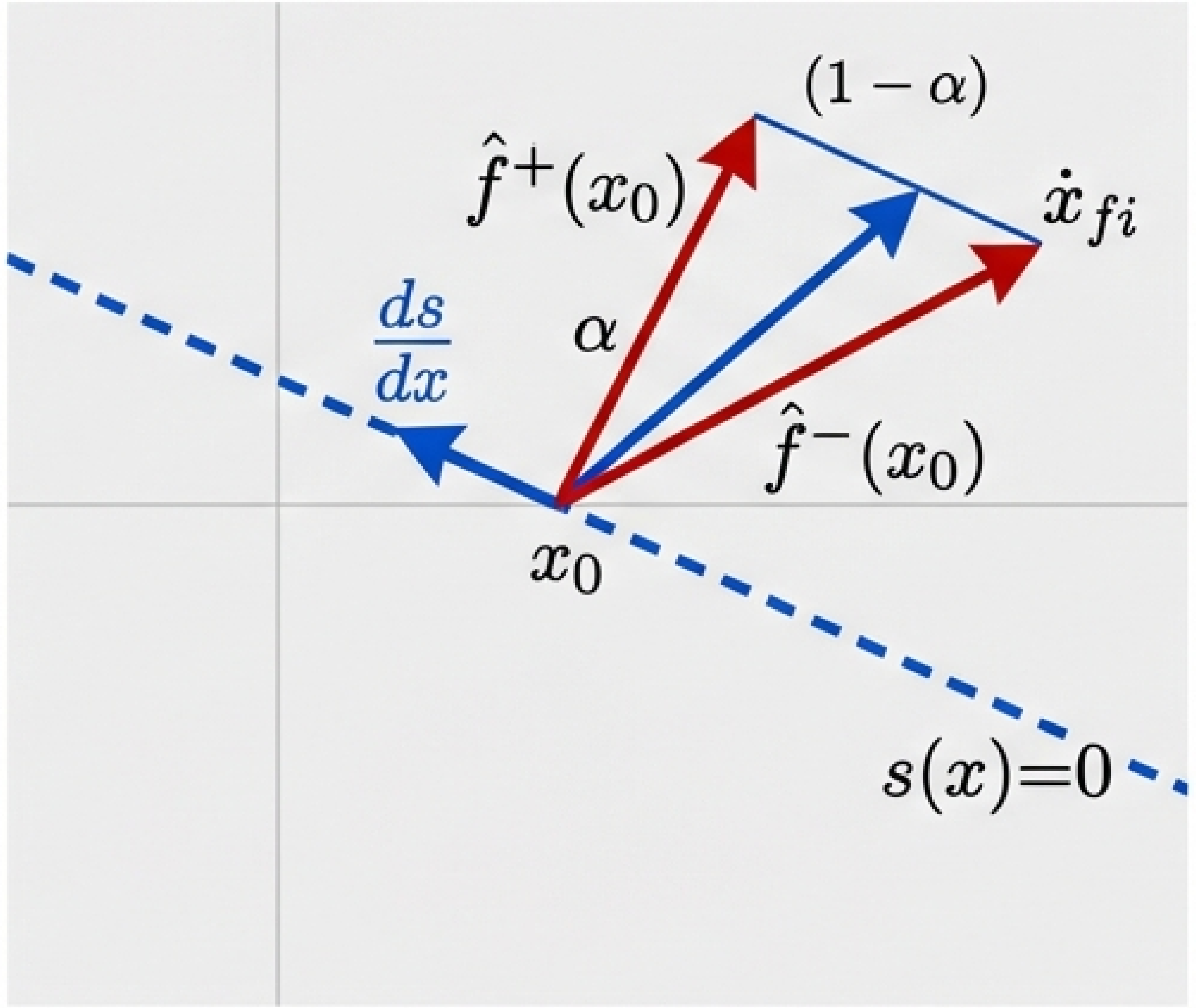
$$\dot{x}_{fi} = \alpha \cdot f^+(x) + (1-\alpha) \cdot f^-(x)$$

Finding Alpha:

The trajectory must be tangent to the surface (orthogonal to gradient).

$$\alpha = \frac{\frac{ds}{dx} f^-(x)}{\frac{ds}{dx}(f^-(x) - f^+(x))}$$

$$0 \leq \alpha \leq 1$$



METHOD 2: EQUIVALENT CONTROL METHOD

Find a continuous input u_{eq} such that the state stays exactly on the manifold ($\dot{s} = 0$).

$$\dot{s} = \frac{ds}{dx}\,\dot{x} = \frac{ds}{dx}\,(f(x) + G(x)u_{eq}) = 0$$

Rearranging: $L_f s(x) + \dfrac{s}{it} + \dfrac{ds}{dx}G(x) \cdot u_{eq} = 0$

$$\boxed{u_{eq} = -\left[\frac{ds}{dx}G(x)\right]^{-1} \cdot L_f s(x)}$$

Equivalent Dynamics:

$$\dot{x}_{eq} = f(x) + G(x)u_{eq}$$

This represents the smooth dynamics on the sliding surface.

APPLICATION: LINEAR OSCILLATOR (DESIGN)

SYSTEM / CONTROLLER

System:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + u$$

Sliding Surface:

$s = x_1 + kx_2 = 0$ **(Straight line)**

Controller:

$$u = -\text{sign}(s) = -\text{sign}(x_1 + kx_2)$$

EXISTENCE ANALYSIS

Check $\dot{s}$:

$$\dot{s} = \dot{x}_1 + k\dot{x}_2 = x_2 - kx_1 - k\cdot\text{sign}(s)$$

Limits Check:

$$\lim_{s\to 0^+} \dot{s} = (1+k^2)x_2 - k < 0$$
$$\lim_{s\to 0^-} \dot{s} = (1+k^2)x_2 + k > 0$$

Conclusion:
Sliding mode exists in domain D defined by bounds on x_2.

LINEAR OSCILLATOR: IDEAL DYNAMICS

On the manifold: $x_1 = -kx_2$

Equivalent Control u_{eq}**:**

Solve $\dot{s} = 0$**:**

$$x_2 - kx_1 + ku_{eq} = 0$$

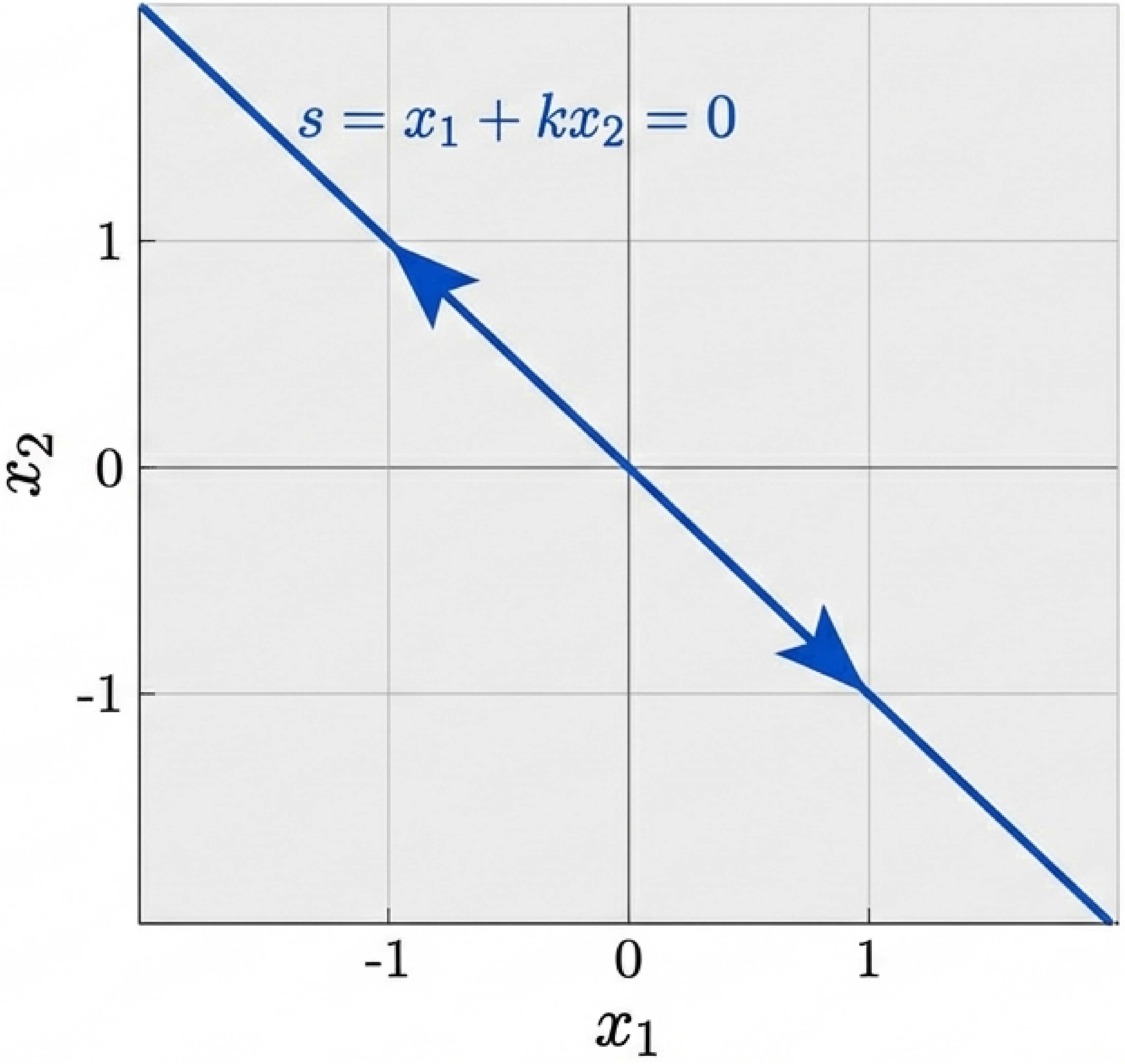
$$u_{eq} = kx_1 - \frac{x_2}{k}$$

Reduced Order Dynamics:

$$\dot{x}_1 = -1/k)x_1$$

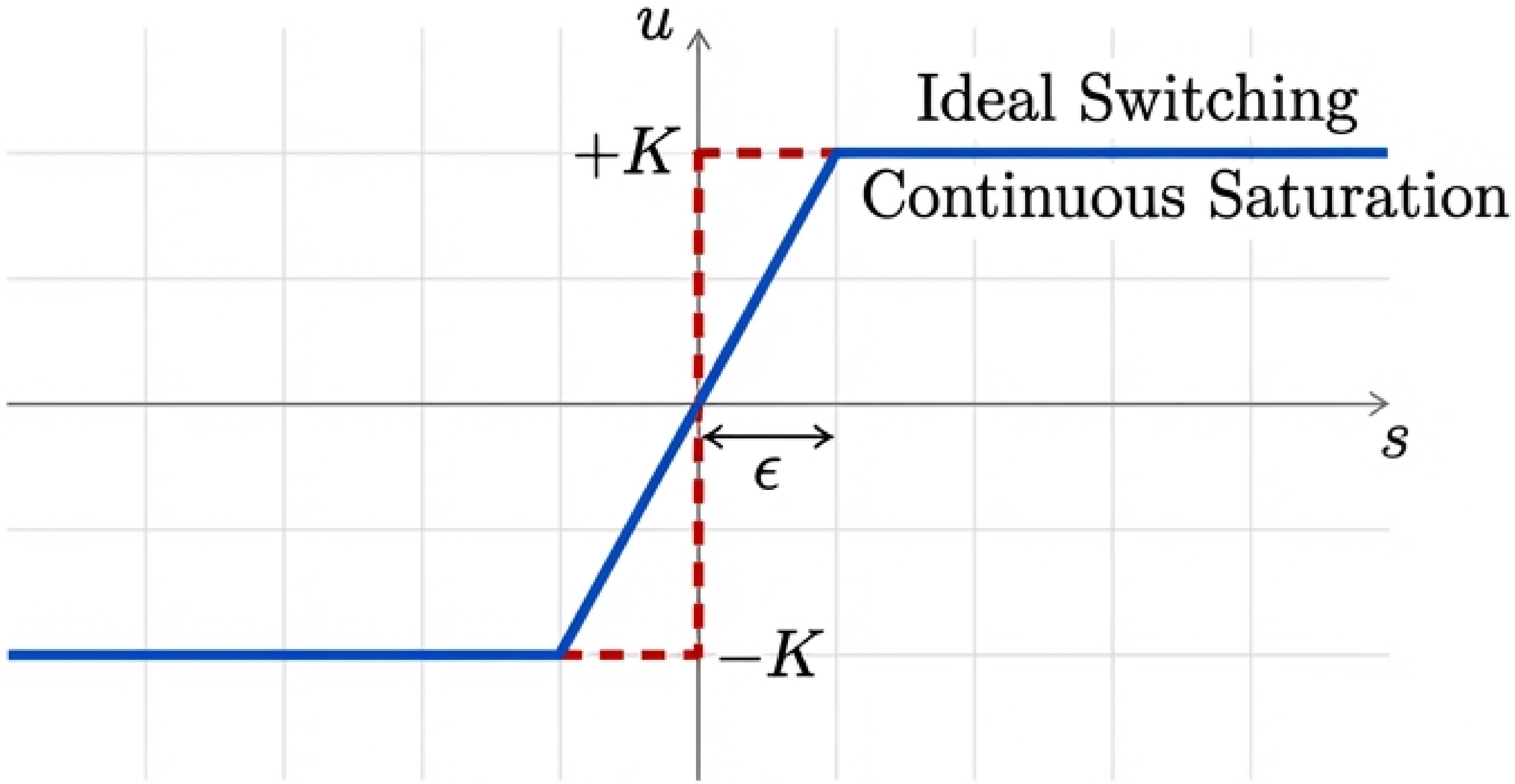
$$\dot{x}_2 = -1/k)x_2$$

Conclusion: System converges to origin exponentially with time constant $1/k$**.**



CONTINUOUS APPROXIMATION

Addressing the Chattering Problem



Control Law:

$$u(x) = -K \cdot sat\left(\frac{s(x)}{\epsilon}\right)$$

Region $|s| > \epsilon$: $u = -K$ (Reaching Phase)

Region $|s| \leq \epsilon$: $u = -\frac{K}{\epsilon}s$ (Linear High-Gain)

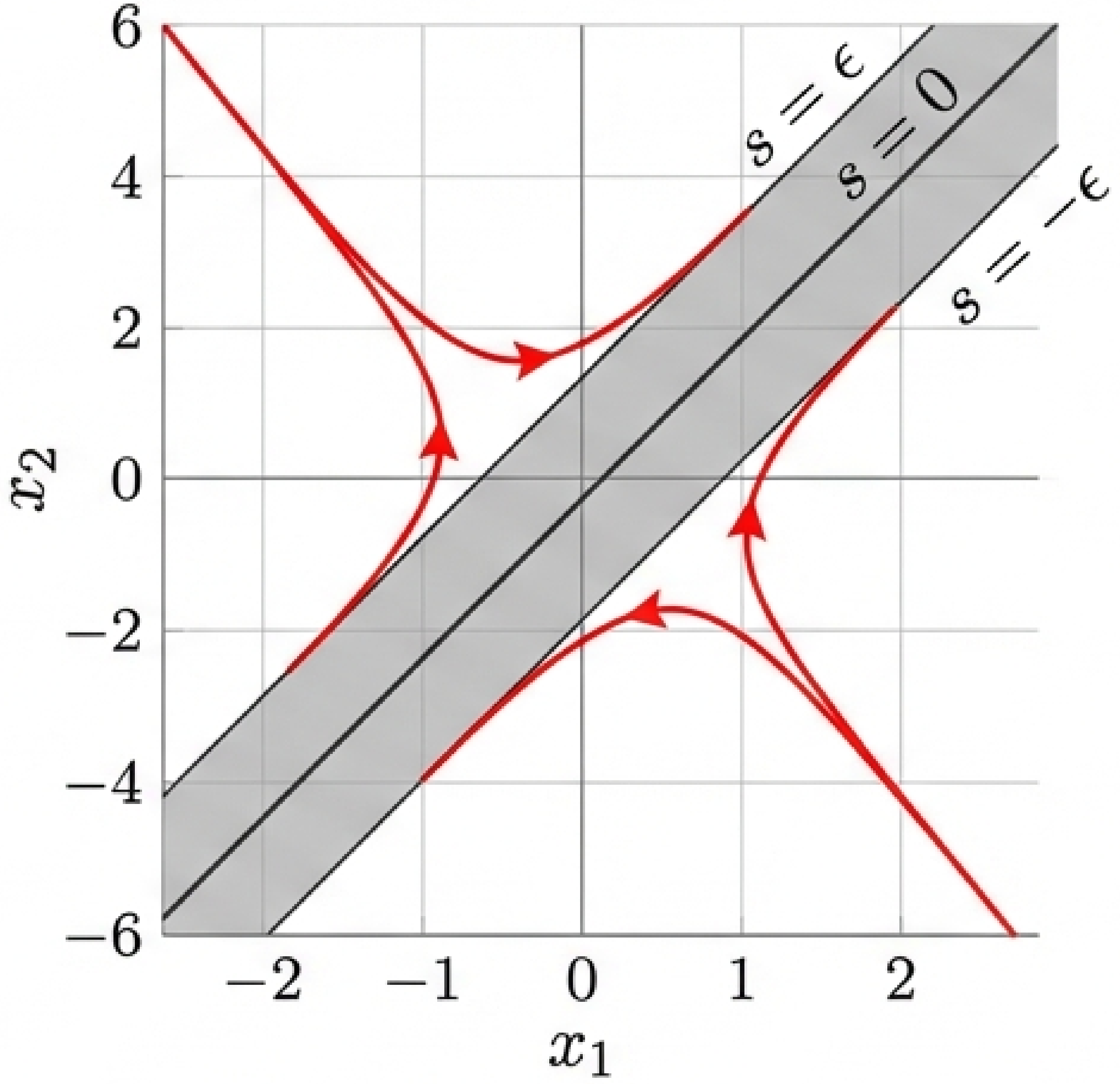
STABILITY: OUTSIDE THE BOUNDARY LAYER

Region: $|s| \geq \epsilon$

Analysis:

- Controller is saturated: $u = -K \cdot \text{sign}(s)$.
- Lyapunov Derivative behaves as in the ideal case:

$$\dot{V} \leq -g_0\beta_0 \ |s|$$



Conclusion:

1. Trajectory reaches the boundary layer $|s| \leq \epsilon$ in finite time.
2. System cannot leave the boundary layer once entered (Positively Invariant).

STABILITY: INSIDE THE BOUNDARY LAYER

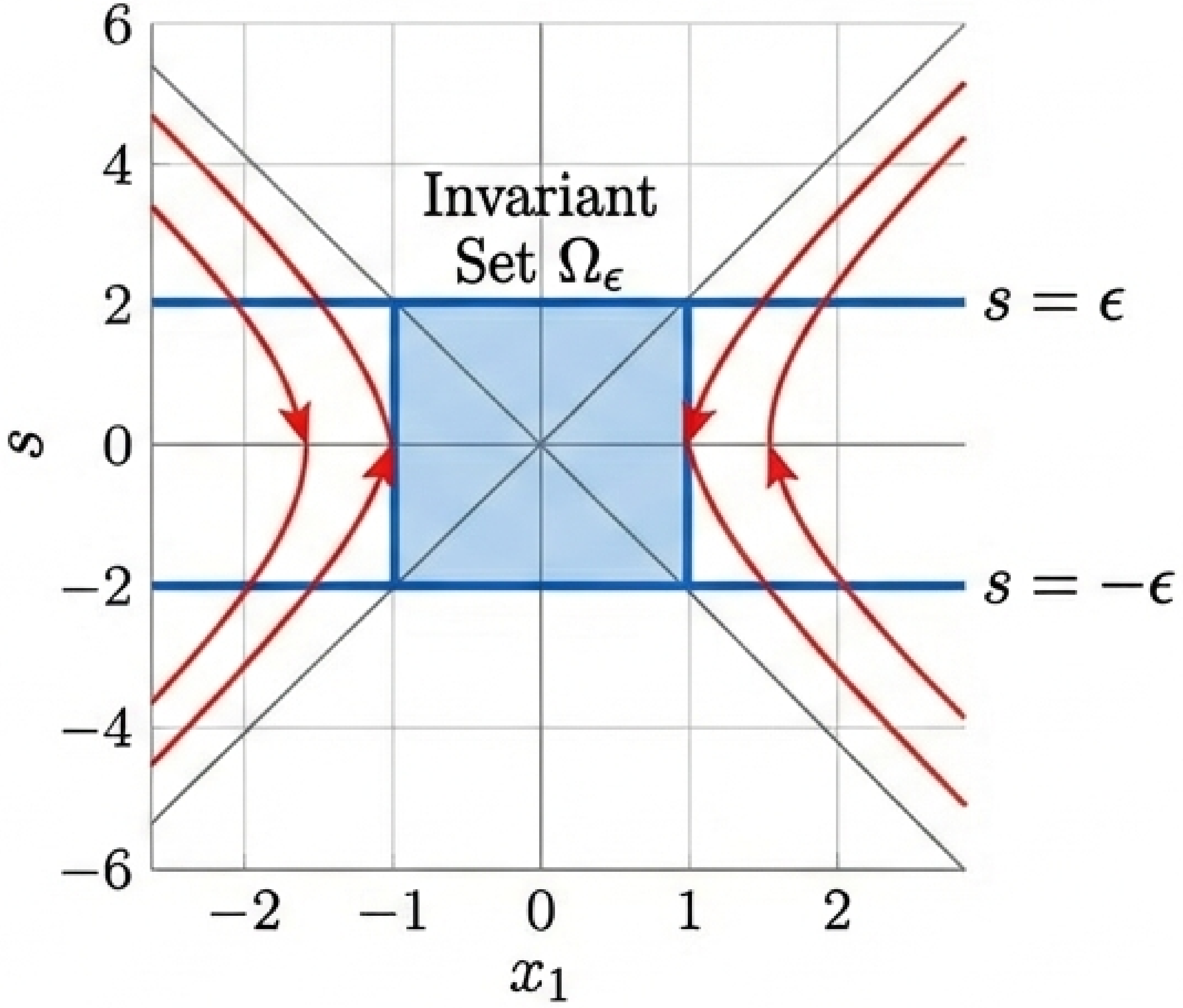
Region: $|s| < \epsilon$

Lyapunov Analysis:

- $\dot{V} < 0$ only holds when $|s| > \epsilon \cdot \frac{\rho\alpha(x)}{\beta\alpha(x)}$.

Input-to-State Stability (ISS):

- Treat 's' as a disturbance on the subsystem.
- $\dot{V}_1 \leq -(a_1 - \theta) \cdot x_1^2$ for $|x_1| \geq |s|/\theta$.



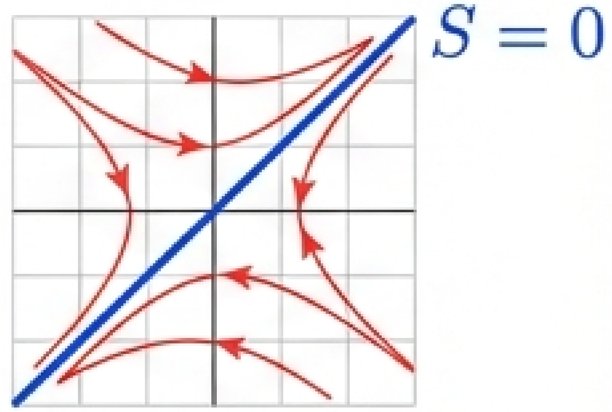
Result (The Invariant Set):

Trajectories converge to the set Ω_ϵ: $\Omega_\epsilon = \{|x_1| \leq |s|/\theta\} \times \{|s| \leq \epsilon\}$

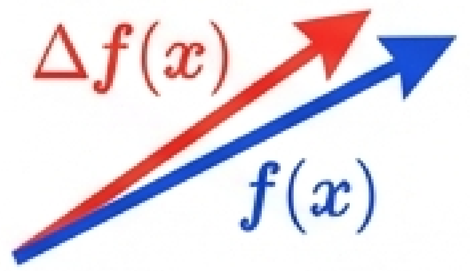
As $\epsilon \longrightarrow 0$, performance recovers ideal sliding mode.

SUMMARY & KEY TAKEAWAYS

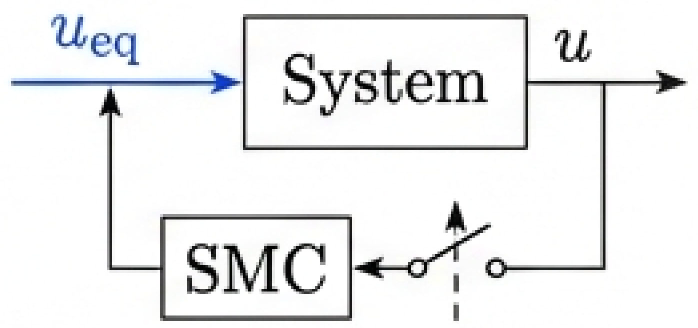
Core Principle: Sliding Mode Control forces the system state onto a user-defined manifold $\mathcal{S}$.



Robustness: Highly robust against matched uncertainties and parameter fluctuations.



Ideal Dynamics: Described by Filippov's method or Equivalent Control (u_{eq}). Results in order reduction.



Implementation: Continuous approximation (Saturation) eliminates chattering.

- **Trade-off:** Convergence to a bounded neighborhood Ω_ε rather than asymptotic stability of the origin.