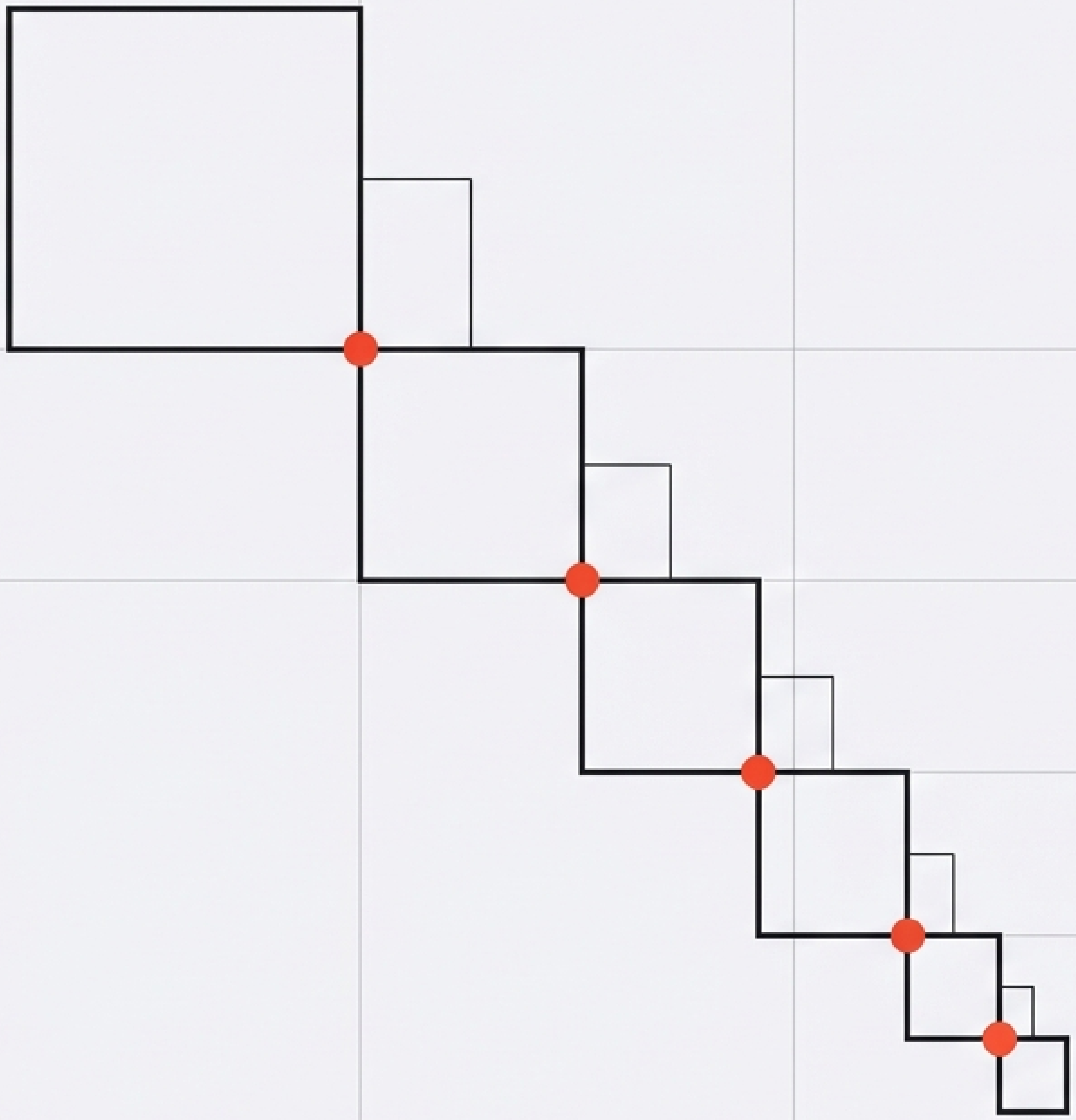
Backstepping Control Design

Recursive Stabilization for Nonlinear Systems in Strict Feedback Form



Source: Lecture Notes: Dynamical Systems (Chapter 7)

Scope of Application: Strict Feedback Form

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2$$

$$\dot{x}_2 = f_2(x_1, x_2) + g_2(x_1, x_2)x_3$$

$$\dots$$

$$\dot{x}_i = f_i(x_1, \dots, x_i) + g_i(x_1, \dots, x_i)x_{i+1}$$

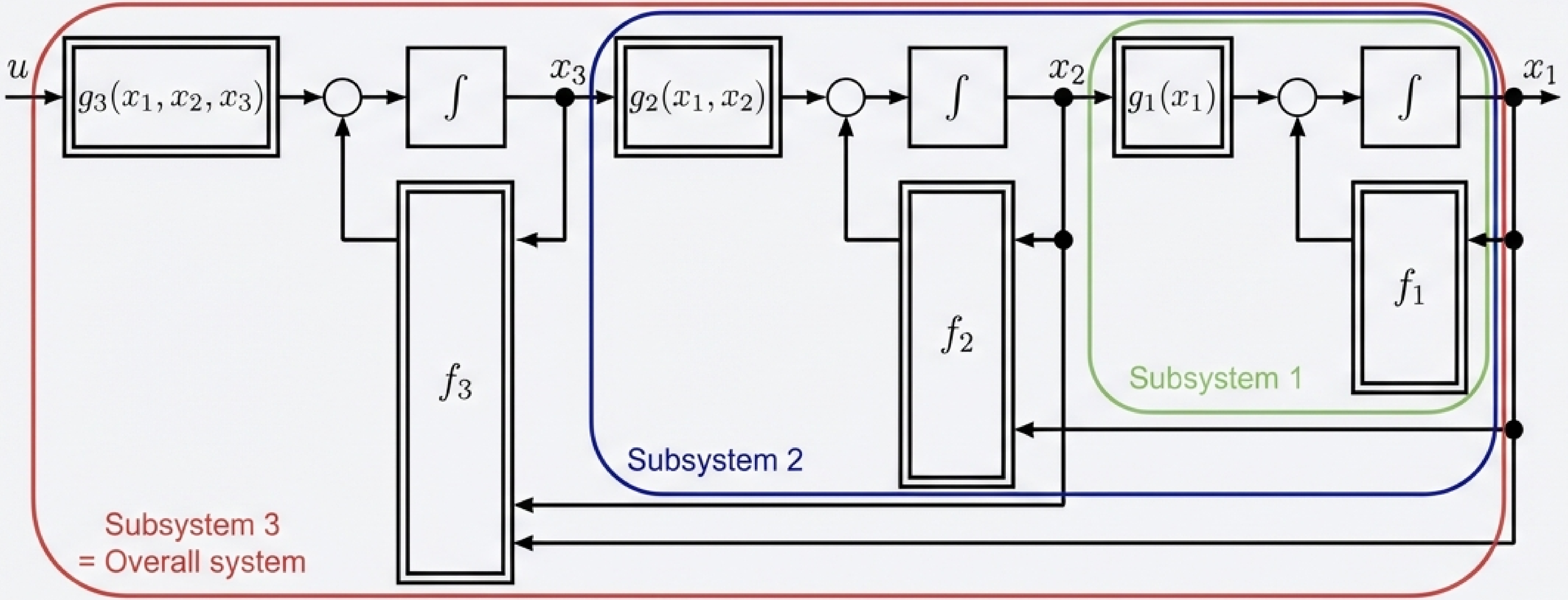
$$\dots$$

$$\dot{x}_n = f_n(x_1, \dots, x_n) + g_n(x_1, \dots, x_n)u$$

Strict Feedback Form Requirements:

1. Triangular structure (state x_i depends only on $x_1 \dots x_i$ and x_{i+1})
2. $f_i(0) = 0$ (Origin is equilibrium)
3. $g_i(\dots) \neq 0$ (Control authority exists)

The General Idea: Recursive Design



Virtual Control
Treat state variable x_{i+1} as the control input for the subsystem x_i.

Coordinate Transformation
Stabilize the error coordinates z_i, not the physical states x_i directly.

Lyapunov Construction
Build a Control Lyapunov Function (CLF) step-by-step for the entire chain.

The Procedure: Coordinate Transformation

The Error Coordinates (z)

$$z_1 := x_1$$

$$z_i := x_i - \alpha_{i-1}(z_1, \ldots, z_{i-1})$$

Virtual Controller from previous step

The Lyapunov Function (V)

$$V_1 = 0.5 * z_1^2$$

$$V_i = V_{i-1} + 0.5 * z_i^2$$

$$V_n = 0.5 * \sum z_i^2$$

We augment the energy function at every step of the recursion.

Step 1: Stabilizing the First Subsystem

System Dynamics

$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2$

Lyapunov Candidate

$V_1 = 0.5 * z_1^2$

$\dot{V}_1 = z_1(f_1(x_1) + g_1(x_1)x_2)$

Virtual Control Design

Set $x_2 = \alpha_1$

Choose α_1 such that $\dot{V}_1 < 0$

Example: $\alpha_1 = -\dfrac{f_1 + c_1 z_1}{g_1}$

Recursive Step i & Final Control Law

The Transformation Link:

$$z_i = x_i - \alpha_{i-1}$$

$$\dot{z}_i = \dot{x}_i - \dot{\alpha}_{i-1}$$

Requires **partial derivatives** (Chain Rule). Source of complexity.

Stabilization Condition:

Find α_i such that $\dot{V}_i \leq 0$

Final Step (n):

The physical input **u** appears.

$$u = \alpha_n(x_1, \ldots, x_n)$$

Result: Explicit control law for Global Asymptotic Stability.

Illustrative Example: 3rd-Order System

System Model:

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = u$$

Objective:

Global stabilization of the origin x=0.

Strategy:

1. Use x_2 to stabilize x_1.
2. Use x_3 to stabilize x_2.
3. Use u to stabilize x_3.

Example Step 1: Base Stabilization

Derivation:

Transformation:

$$z_1 = x_1$$

Lyapunov:

$$V_1 = 0.5 \cdot z_1^2$$

Derivative:

$$\dot{V}_1 = z_1(x_1^2 - x_1^3 + x_2)$$

Design:

Virtual Control (α_1):

We treat $\mathbf{x_2}$ as the control input α_1.

To make $\dot{V}_1$ negative definite, we choose:

$$\alpha_1 = -x_1^2 - x_1$$

(Note: derived to cancel x1 terms and damp)

Result:

$\dot{V}_1 = -z_1^4$ (Negative Definite)

Example Step 2: Intermediate Stabilization

Transformation:

$$z_2 = x_2 - \alpha_1$$

Dynamics:

$$\dot{z}_2 = x_3 - \dot{\alpha}_1$$

Calculation Detail:

$$\dot{\alpha}_1 = (3z_1^2 - 2z_1 - 1)(z_2 - z_1)$$

Lyapunov Function:

$$V_2 = V_1 + 0.5 * z_2^2$$

Virtual Control (α_2):

$$\alpha_2 = -z_1 - z_2 + \dot{\alpha}_1$$

Result:

$$\dot{V}_2 = -z_1^4 - z_2^2$$

Example Step 3: Deriving the Control Law

Derivation:

- Transformation: z_3 = x_3 - α_2
- Total Lyapunov: V_3 = V_2 + 0.5 * z_3^2

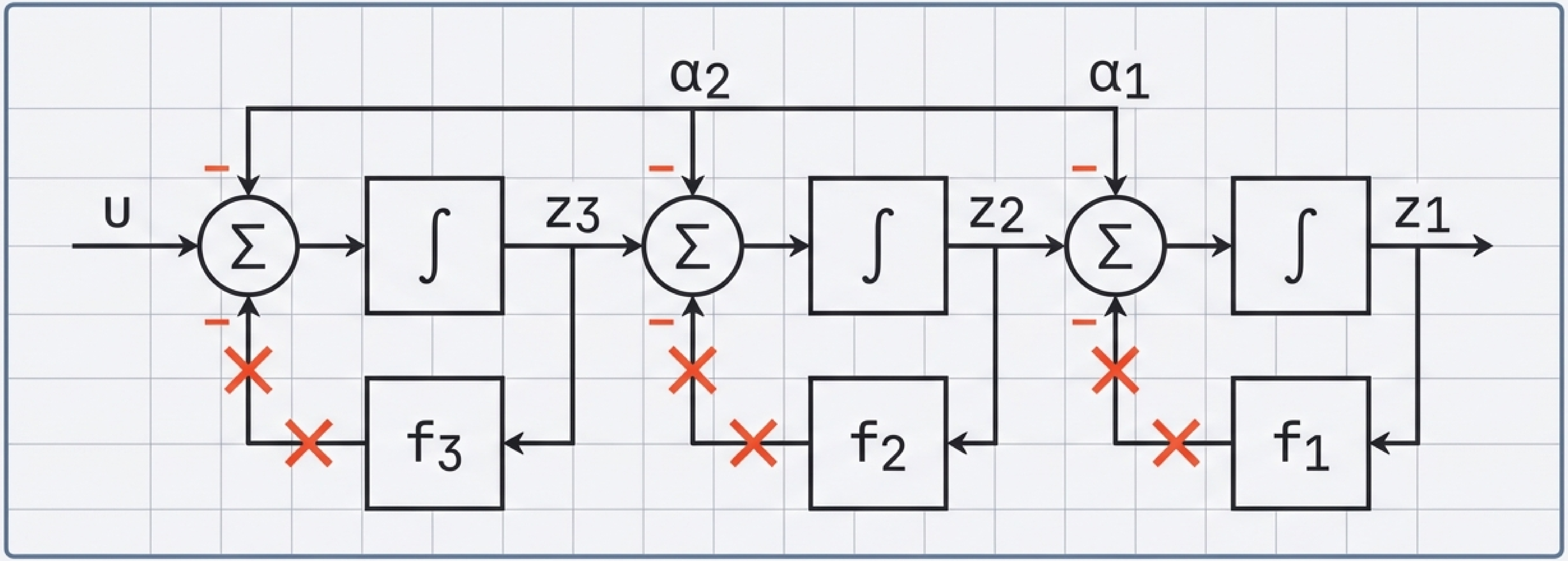
Design:

The Physical Control Input u

$$u = -z_2 - z_3 - (3z_1^2 - 2z_1)(z_2 - z_1) + (6z_1 - 2)(z_2 - z_1)^2 + (3z_1^2 - 2z_1 - 2)(z_3 - z_1 - z_2)$$

This explicit control law **u(x)** guarantees Global Asymptotic Stability.

Structure of the Transformed System



In z-coordinates, the complex nonlinear system behaves like a linear chain of integrators. The control law u cancels the nonlinearities.

Robust Backstepping: The Problem

Standard Backstepping assumes exact knowledge of $f(x)$ and $g(x)$.

In ideal conditions, the system's non-linearities, represented by the functions $f(x)$ and $g(x)$, are considered perfectly known, allowing for precise mathematical cancellation in the control law derivation.

This simplification enables straightforward tracking and stabilization.

Real-World Scenario: Uncertainty

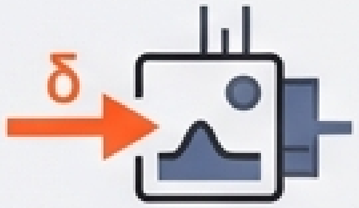
$$\dot{x}_2 = f(x) + g(x)u + \delta$$

Assumption: Unknown but bounded disturbance.

$$|\delta| \leq \delta_{max}$$

Challenge: Since δ is unknown, we cannot mathematically cancel it in the control law u.

This uncertainty prevents the direct compensation of the disturbance, leading to potential instability or performance degradation without a robust control strategy.



Disturbance Entry

Robust Design Strategy: Domination

Analysis

$$\dot{V} = -k\|z\|^2 + z_2\delta$$

Disturbance

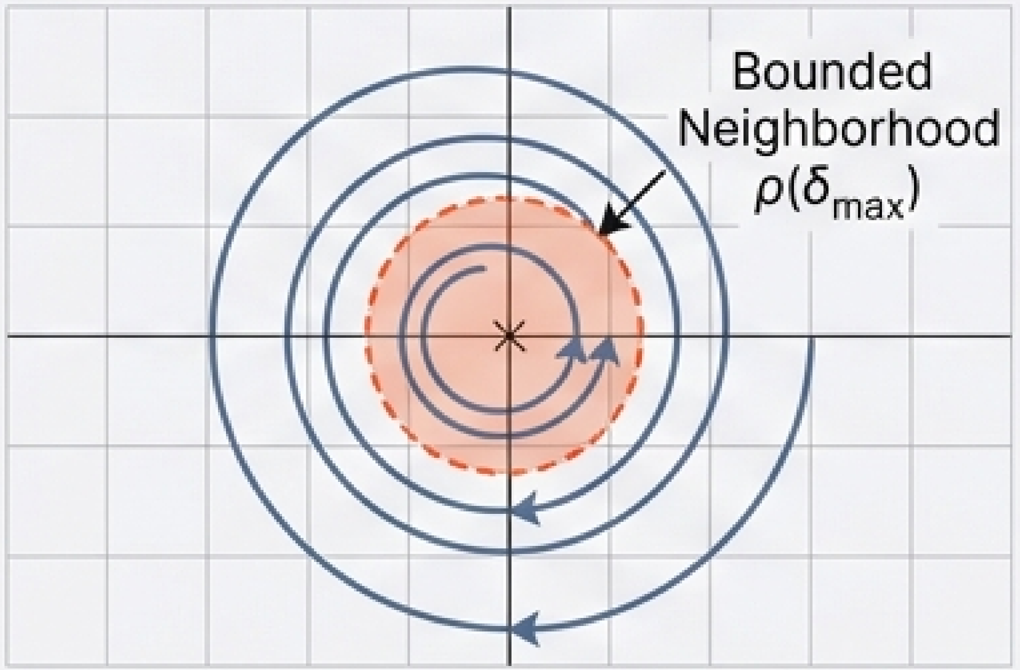
Domination (Young's Inequality)

$$z_2\delta \leq 0.5\,z_2^2 + 0.5\,\delta^2$$

Result

$$\dot{V} \leq -k_0\|z\|^2 + 0.5\,\delta^2$$

System is Input-to-State Stable (ISS). Trajectories do not go to zero, but converge to a bounded neighborhood $\rho(\delta_{max})$.



Evaluation: Pros and Cons

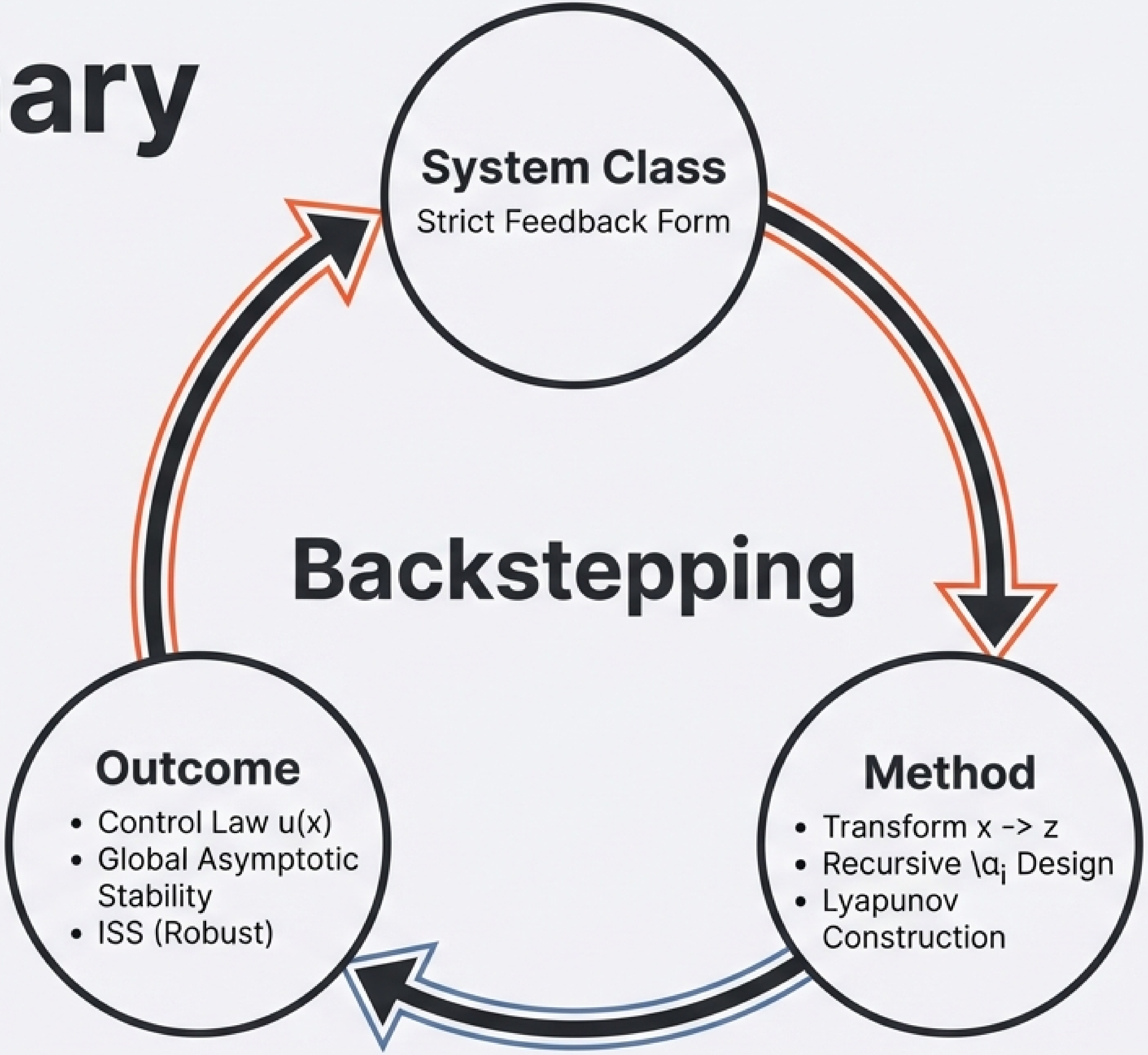
Advantages

- Systematic, recursive recipe.
- Handles Strict Feedback nonlinearities.
- Achieves Global Stability (unlike linearization).
- Flexible design.

Disadvantages

- Explosion of Terms: Analytic complexity grows rapidly with system order n.
- Requires precise model (unless using robust/adaptive variants).
- High control effort possible.

Summary


System Class
Strict Feedback Form
Backstepping
Method
• Transform x -> z
• Recursive \α_i Design
• Lyapunov Construction
Outcome
• Control Law u(x)
• Global Asymptotic Stability
• ISS (Robust)