2. Lyapunov Stability

1: Requirements

Requirements	
1) autonomous	$\dot{\underline{x}} = f(\underline{x}, t)$
2) $\exists$ unique sol ?	Existence of (local / global) unique solution $\rightarrow$ $\underline{f}(\underline{x}, t)$ must locally / globally Lipschitz continous.
Sufficient condition for Lip-Continous	1) If $\underline{f}(\underline{x}, t)$ is continous and continously differentiable $\Rightarrow$ $\underline{f}(\underline{x}, t)$ is locally Lipschitz continous 2) If additionally all $\frac{\partial f_i}{\partial x_j}$ are bounded: $\Rightarrow$ $f(x, t)$ is globally Lipschitz continous

$y = x^2$
$\rightarrow y' = 2x$
local.L.C

$y = 1$
$\rightarrow y' = 0$.
glo.L.C

2. Lyapunov Stability

2: Lyapunov method – Direct method EP, V(x) pdf ?, $\dot{V}(x)$ n(s)df ?

Lyapunov's direct method

pdf 가 아닐 경우?
→ 다음장 LaSalle's

1) EP: $\underline{x}^* \stackrel{!}{=} 0$ (else: state transformation)

2) Lya. function $V(x)$ is positive definite (pdf) and continuously differentiable (conti. diff)

cf) energy-like ...

$$V(x) \text{ pdf} \iff V(x) = \begin{cases} = 0, & \text{for } \underline{x} = 0 \\ > 0, & \underline{x} \neq 0 \end{cases}$$

3) $\dot{V}(\underline{x})$ is negative (semi) definite ~> deriv. respect to time

$$V(x) \text{ ndf} \iff V(x) = \begin{cases} = 0, & \text{for } \underline{x} = 0 \\ < 0, & \underline{x} \neq 0 \end{cases}$$

$$V(x) \text{ nsdf} \iff V(x) = \begin{cases} = 0, & \text{for } \underline{x} = 0 \\ \leq 0, & \underline{x} \neq 0 \end{cases}$$

Stability Conclusions

$V(x)$	$\dot{V}(x)$	**Conclusion**
pdf	nsdf	stable
pdf	ndf	asymptotically stable
globally pdf, radially unbounded	globally nsdf	globally stable
globally pdf, radially unbounded	globally ndf	globally asy. stable

must
If $\|x\| \to \infty$, $V(x) \to \infty$

Considering time dependency:

→ Stability conclusion is uniform, if $V(\underline{x}, t)$ is decresent

→ If no explicit dependency on time t ⇒ $V(\underline{x})$ decresent

2. Lyapunov Stability

2: Lyapunov method – LaSalle's Invariance Principle

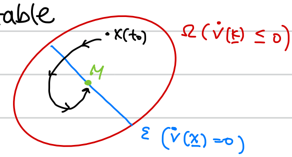
LaSalle's Invariance Prinzip

$\Rightarrow$ applicable if $V(x)$ is not pdf

1) $V(x)$ is cont. diff
2) $\dot{V}(x) \leq 0$ for all $\underline{x}$ in invariant set Ω
3) $\mathcal{E} = \{ \underline{x} \in \Omega \mid \dot{V}(\underline{x}) = 0 \}$

$\Rightarrow$ Solution of $\dot{\underline{x}} = \underline{f}(\underline{x})$ converges to (largest) invariant set $\mathcal{M} \subseteq \mathcal{E}$

cf) If $\mathcal{M}$ only contains EP $\Rightarrow$ asymptotically stable



Corollaries

$\Rightarrow$ show asymptotical stability if $\dot{V}(x)$ is only nsdf

- **Barbashin (local)**
  - $V(x)$ is pdf on B_ε
  - $\dot{V}(x) \leq 0$ on B_ε
  - $S = \{ \underline{x} \in B_\varepsilon \mid \dot{V}(\underline{x}) = 0 \}$

  ☆ EP $\underline{x}^* = \underline{0}$ is loc. asym. stable if only $\underline{x}(t) \equiv 0$ may remain in S

- **Krasovskii (global)**
  - $V(x)$ is rapidally unbounded
  - $V(x)$ is globally pdf on $\mathbb{R}^n$
  - $\dot{V}(x) \leq 0$ globally on $\mathbb{R}^n$
  - $S = \{ \underline{x} \in \mathbb{R}^n \mid \dot{V}(\underline{x}) = 0 \}$

  ☆ EP $\underline{x}^* = \underline{0}$ is glob. asym. stable if only $\underline{x}(t) \equiv 0$ may remain in S

2. Lyapunov Stability

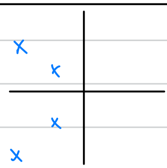
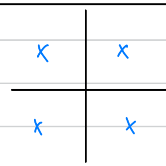
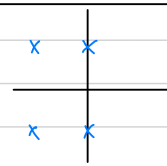
2: Lyapunov method – Lyapunov's Indirect Method (time-invariant)

Requirements

1) Dynamics $\dot{\underline{x}} = \underline{f}(\underline{x})$, with $\underline{f}(\underline{x})$ continuously differentiable
2) Arbitary EPs $\underline{x}^*$

Procedure

1) Linearize: $\dot{\underline{x}} = \underline{f}(\underline{x}) \rightarrow \dot{\underline{x}} = A\underline{x}$, with $A = \left[\frac{\partial \underline{f}}{\partial \underline{x}}\right]\Big|_{\underline{x} = \underline{x}^*}$
2) compute Eigenvalues λ_i : $\det(A - \lambda I) \overset{!}{=} 0$
3) Evaluation (only holds locally)

$\forall i : Re\{\lambda_i\} < 0$	$\exists i : Re\{\lambda_i\} > 0$	$\exists i : Re\{\lambda_i\} = 0$ $\forall i : Re\{\lambda_i\} \leq 0$
asym. stable	unstable	no evaluation
		

2. Lyapunov Stability
3: Linear vs Non-linear Systems

Lyap. Stability of LTI	Sufficient conditions for asymptotic stability 1) $V(x)$ is pdf and cont. diff 2) $\dot{V}(x)$ is ndf
Strategy for LTI : $\dot{\underline{x}} = A\underline{x}$	• Choose $V(\underline{x}) = \underline{x}^T P \underline{x}$, with $P \succ 0$ (pd matrix) • $\dot{V}(\underline{x}) = \dot{\underline{x}}^T P \underline{x} + \underline{x}^T P \dot{\underline{x}} = \underline{x}^T (A^T P + PA) \underline{x}$
cf) Lyapunov's Direct Method for LTI	If there exists $P \succ 0$, $Q \succ 0$, s.t. $A^T P + PA = -Q$, $\Rightarrow$ EP $\underline{x}^* = 0$ is globally asymptotically stable.

Stability Analysis using Eigenvalues

• LTI Systems : $\dot{\underline{x}} = A\underline{x}$

$\Rightarrow$ Compute eigenvalues $\lambda_i(A)$: $\det(\lambda I - A) \overset{!}{=} 0$

$\forall_i : Re\{\lambda_i\} < 0$	$\exists_i : Re\{\lambda_i\} = 0$ $\forall_i : Re\{\lambda_i\} \leq 0$	$\exists_i : Re\{\lambda_i\} > 0$
asym. stable	stable	unstable

• LTV Systems : $\dot{\underline{x}} = A(t)\underline{x}$

EP $\underline{x}^* = 0$ is uniformly globally asymptotically stable if

1) $\forall_i : Re\{\lambda_i(A(t) + A(t)^T)\} < 0$ for $\forall t$

OR

2) $\forall_i : Re\{\lambda_i(A(t))\} < 0$ for $\forall t$, and $\int_0^\infty A(t)^T A(t)\, dt < \infty$.

2. Lyapunov Stability
3 : Linear vs Non-linear Systems

Lyapunov Functions for Nonlinear Systems

$\Rightarrow$ there exists no generally suitable Lyapunov function $V(x)$

Commonly used candidate functions:

- Energy function (resulting from energy balance analysis)
- $V(\underline{x}) = \underline{x}^T \underline{x}$
- $V(\underline{x}) = \underline{x}^T P \underline{x}$, with $P \succ 0$
- $V(\underline{x}) = \sin^2(\underline{x}^T \underline{x})$
- Backstepping : $V(\underline{x}) = \frac{1}{2} \sum_{i=1}^{n} e_i^2$ with e_i ← tracking error of each systems
- Sliding Mode Control : $V(\underline{x}) = \frac{1}{2} \underline{s}^T \underline{s}$ with $\underline{s}$ ← sliding variables

2. Lyapunov Stability
4: Domain of Attraction

※ For locally stable EPs of the System $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$, $\underline{x}(t_0) = \underline{x}_0$

DoA : Domain of Attraction	The DoA $A(\underline{x}^*)$ of EP $\underline{x}^*$ is the set of all initial states $\underline{x}_0$, for which the resulting trajectory converges to the EP.
	DoA is open, coherent, invariant set.
	(Its boundary invariant, and consists of trajectory system
DoA Problem	exact computation is difficult/impossible in most cases
Strategy	→ Use Lyapunov function $V(\underline{x})$ to compute $\mathcal{E}_c \subseteq A(\underline{x}^*)$
	1) EP $\underline{x}^*$ is asym. stable according to $V(\underline{x})$
	2) Define $\mathcal{V} = \{\underline{x}^*\} \cup \{\underline{x} \mid V(\underline{x}) > 0,\ \dot{V}(\underline{x}) < 0\}$
	3) Define $\mathcal{E}_c = \{\underline{x} \mid V(\underline{x}) \leq c\}$
	$\Rightarrow$ $\mathcal{E}_c \subseteq A(\underline{x}^*)$, if $\mathcal{E}_c \in \mathcal{V}$ and $\mathcal{E}_c$ is bounded.

2. Lyapunov Stability

5: Lyapunov Based Controller Design

*	Lyapunov theory can be used to design stabilizing controllers
non-autonomous system	$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u})$
Lyapunov Based Controller	1) Choose candidate $V(\underline{x})$ 2) Compute $\dot{V}(\underline{x}, \underline{u}) = \frac{\partial V}{\partial \underline{x}} \underline{f}(\underline{x}, \underline{u})$ 3) Find feedback law $\underline{u} = \underline{k}(\underline{x})$, such that $V(\underline{x})$, $\dot{V}(\underline{x}, \underline{k}(\underline{x}))$ fulfill Lyapunov stability conditions $\Rightarrow$ autonomous system $\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{k}(\underline{x}))$ is $(\cdots)$ stable.