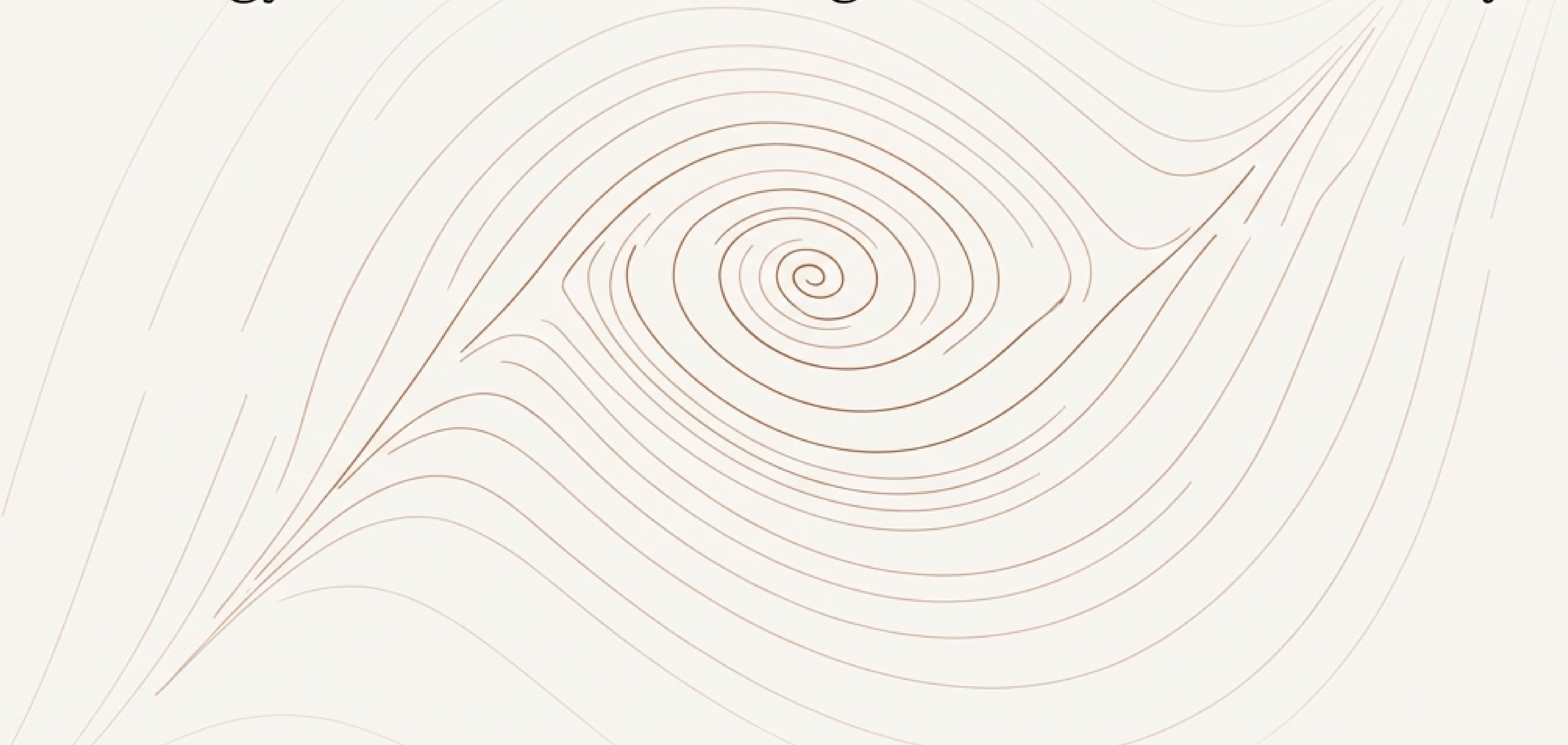
# A Deep Dive into Passivity

## From Energy Balance to the Design of Stable Nonlinear Systems



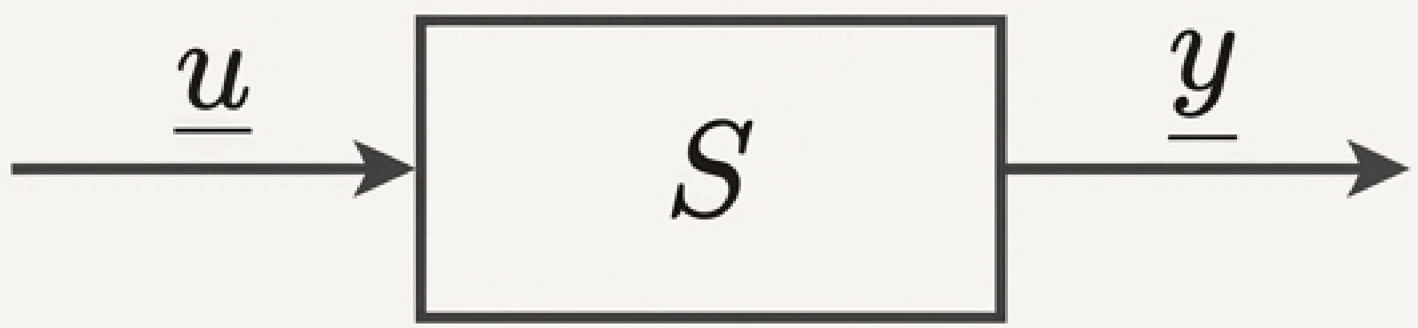
Based on Lecture Notes: Dynamical Systems, Technical University of Munich

# Passivity analyzes systems through the lens of energy flow.

We consider a general time-invariant dynamical system, S, with an input u and an output y. The system is described by the state space model:

- $\dot{x} = f(x, u)$
- $y = h(x, u)$

The concepts of dissipativity and passivity provide a framework to analyze the flow of "energy" into and out of the system, which directly informs its stability properties.

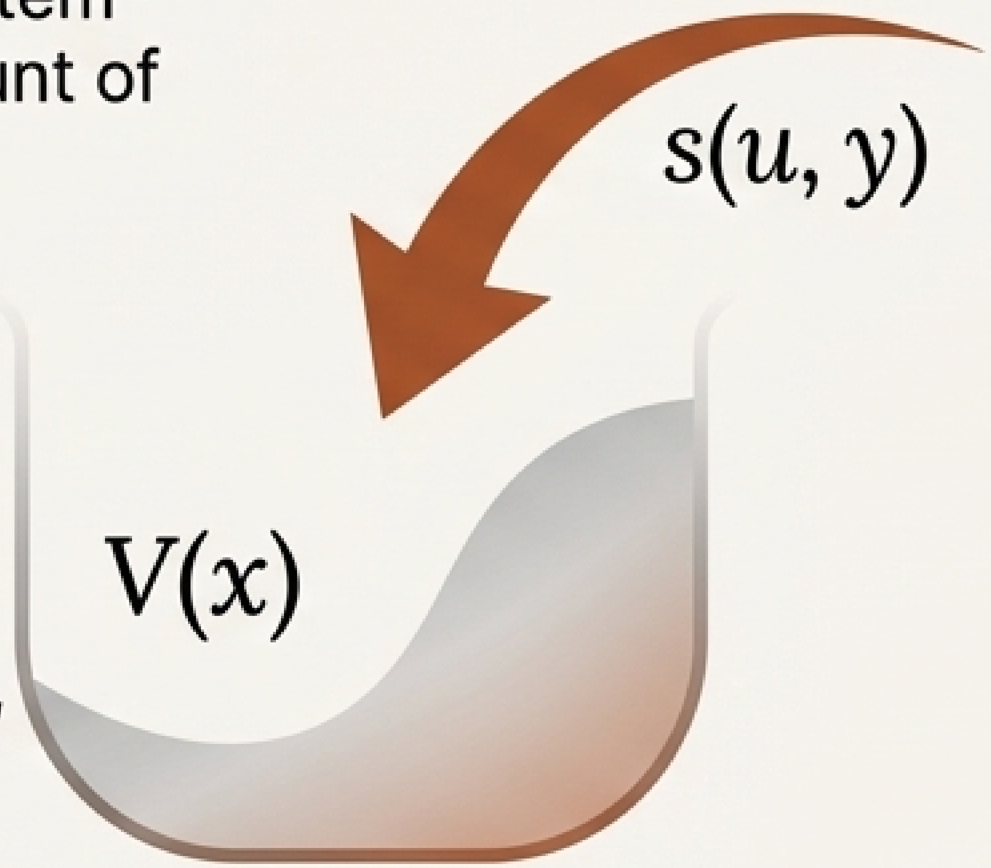


# The foundation of passivity is the principle of dissipativity.

Dissipativity formalizes the energy balance of a system. The energy stored within a system cannot increase by more than the amount of energy supplied to it.

This is defined by two key concepts:

- Storage Function, **V(x)**: Represents the energy stored within the system's state **x**. It must be a positive semidefinite function (psdf), meaning **V(x)** $\geq 0$.




- Supply Rate, **s(u, y)**: A real-valued function representing the instantaneous power flowing into the system.

The fundamental principle is:

**[Net energy supplied] + [Initial stored energy] ≥ [Final stored energy]**

# A system is dissipative if the supplied energy accounts for any increase in stored energy.

A system S is dissipative with respect to a supply rate $s(u, y)$ if there exists a positive semidefinite (psdf) storage function $V(x)$ such that the following inequality holds:

## Integral Form

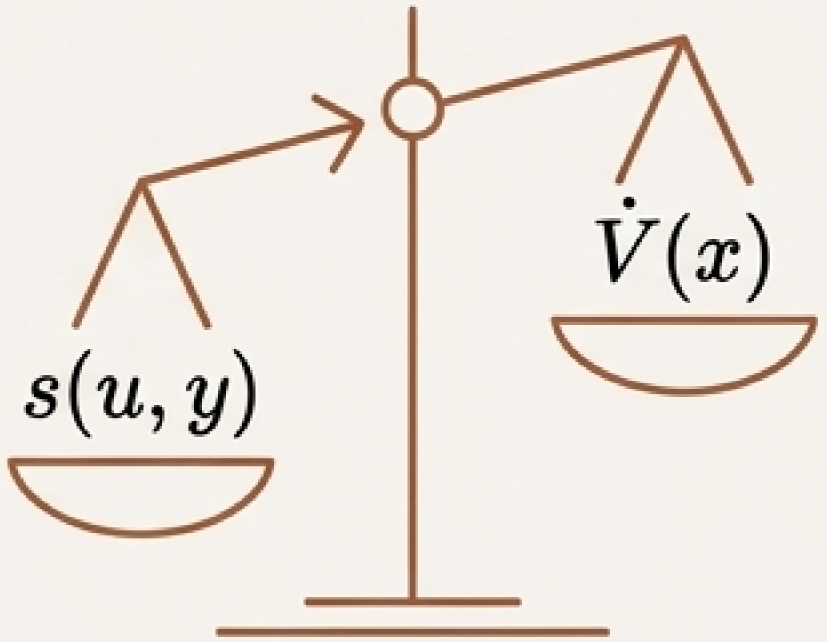
The total energy supplied over an interval is greater than or equal to the net increase in stored energy.

$$\int_0^t s(u, y) d\tau + V(x(0)) \geq V(x(t))$$

## Differential Form

The instantaneous power supplied is greater than or equal to the rate of change of stored energy.

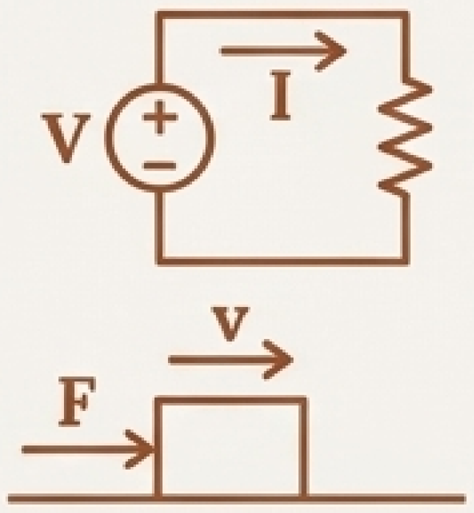
$$s(u, y) \geq \dot{V}(x(t))$$



# Passivity is a special, physically meaningful case of dissipativity.

A system is **passive** if it is dissipative with respect to the specific supply rate $s(\mathbf{u}, \mathbf{y}) = \mathbf{y}^\top \mathbf{u}$.

- This form is physically significant, often representing power in electrical (voltage × current) and mechanical (force × velocity) systems.
- This definition requires $\dim(\mathbf{u}) = \dim(\mathbf{y})$.




Differential Passivity Inequality:

$$\mathbf{y}^\top \mathbf{u} \geq \dot{V}(\mathbf{x}(t))$$

A system is defined as **Lossless** if this inequality holds with equality:

$$\mathbf{y}^\top \mathbf{u} = \dot{V}(\mathbf{x}(t))$$

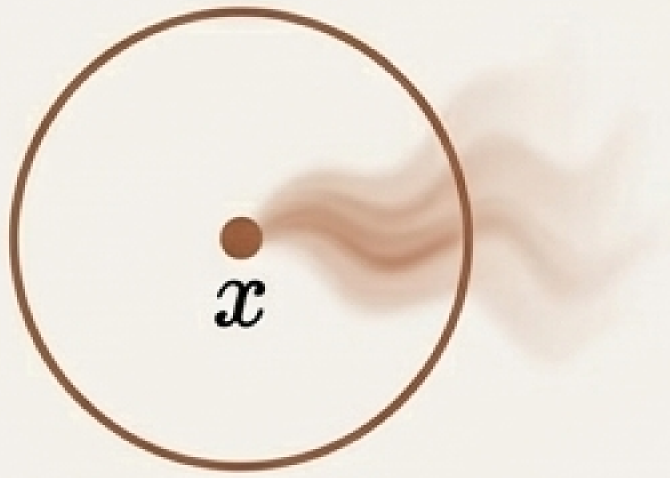
# Stricter forms of passivity provide stronger guarantees on energy dissipation.

We can define stricter forms of passivity that require the system to actively dissipate energy, leading to stronger stability properties.

## State Strictly Passive

Dissipation is guaranteed as a function of the state **x**. The function $\Psi(\cdot)$ must be positive definite (pdf).

$$\dot{V}(x(t)) \leq y^{\mathrm{T}} u - \Psi(x(t))$$




## Output Strictly Passive

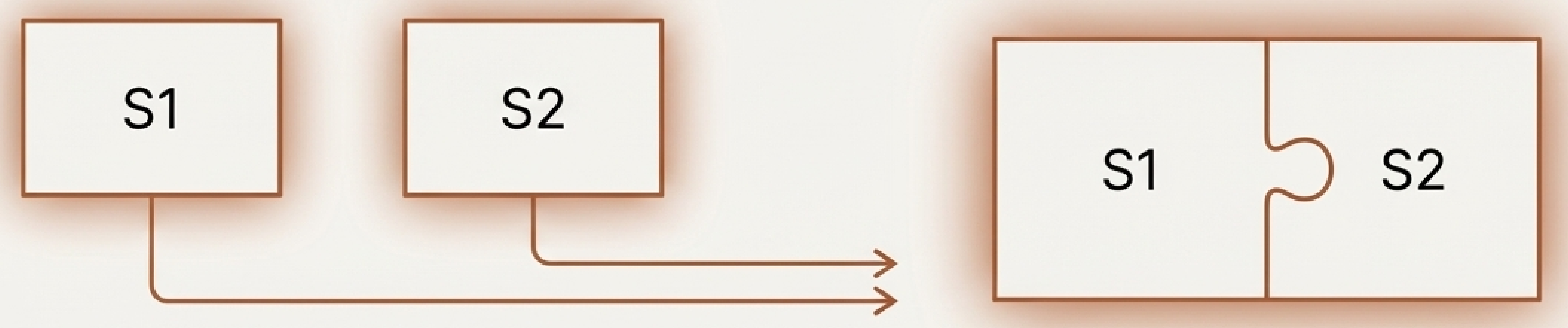
Dissipation is guaranteed as a function of the output **y**. The function $\mathbf{y}^{\mathrm{T}}\rho(\mathbf{y})$ must be positive definite (pdf).

$$\dot{V}(x(t)) \leq y^{\mathrm{T}} u - y^{\mathrm{T}}\rho(y)$$

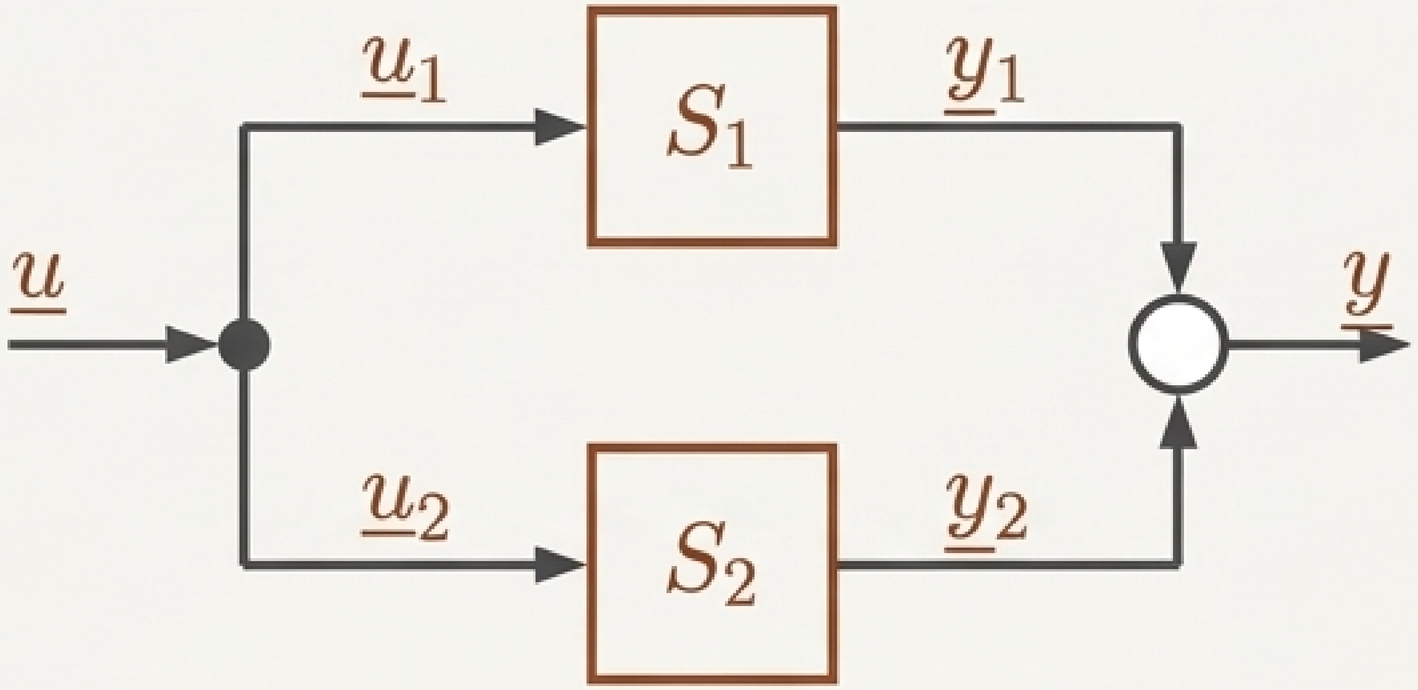
# The key advantage of passivity: it is preserved under interconnection.

This is the most powerful property of passive systems for control design. It enables a **modular approach: if you build a complex system from passive components, the entire interconnected system is guaranteed to be passive.**

Theorem: If systems S1 and S2 are passive, then their parallel and feedback interconnections are also passive.

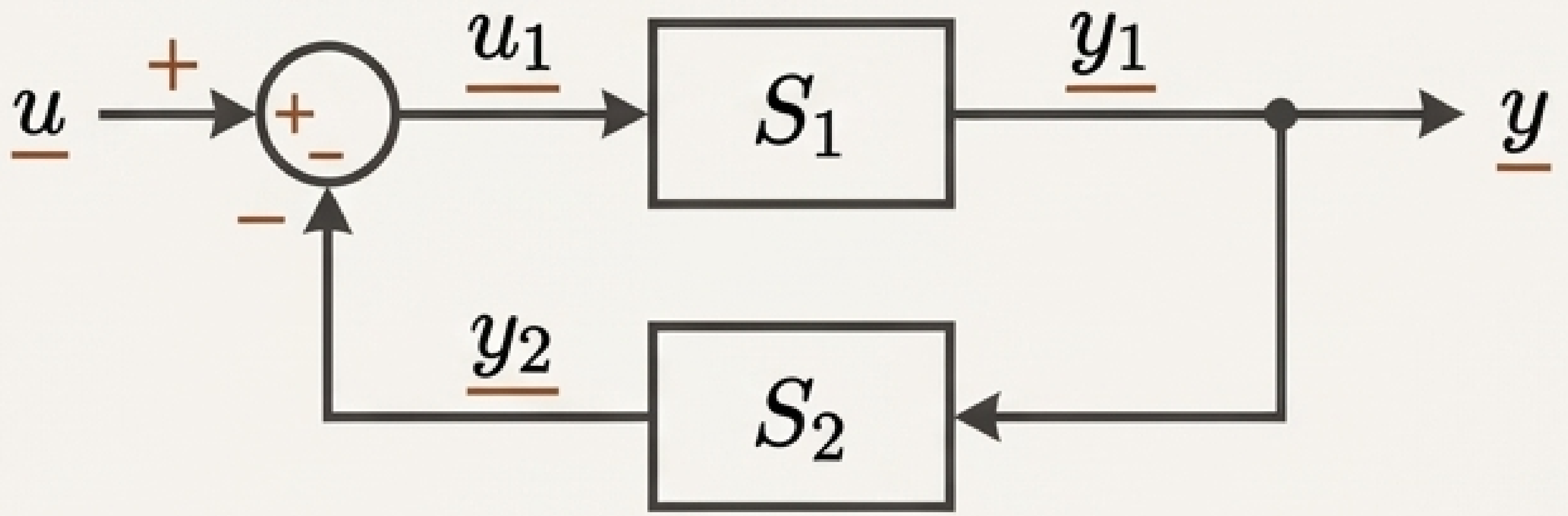


# A parallel combination of passive systems is passive.




- For two passive systems, $S_1$ and $S_2$, connected in parallel:
  - Inputs & Outputs:
    - $u = u_1 = u_2$
    - $y = y_1 + y_2$
  - Storage Function: The total stored energy is the sum of the individual energies.
    - $V_{\text{total}}(x) = V_1(x_1) + V_2(x_2)$
  - Proof Logic: The total supply rate for the interconnected system is $y^T u = (y_1 + y_2)^T u = y_1^T u_1 + y_2^T$. Since each subsystem satisfies the passivity inequality, their sum does too, proving the overall system is passive.

# A negative feedback combination of passive systems is passive.




For two passive systems, S1 and S2, in a negative feedback loop:

- **Inputs & Outputs**:
  - $u = u_1 + y_2$
  - $y = y_1 = u_2$
- **Storage Function**: The total stored energy remains the sum of the parts.
  - $V_{total}(x) = V_1(x_1) + V_2(x_2)$
- **Proof Logic**: The total supply rate is $y^T u = y_1^T(u_1+y_2) = y_1^T u_1 + y_1^T y_2 = y_1^T u_1 + u_2^T y_2$. The sum of individual supply rates again satisfies the passivity inequality for the composite system.

# Passivity provides a direct path to proving Lyapunov stability.

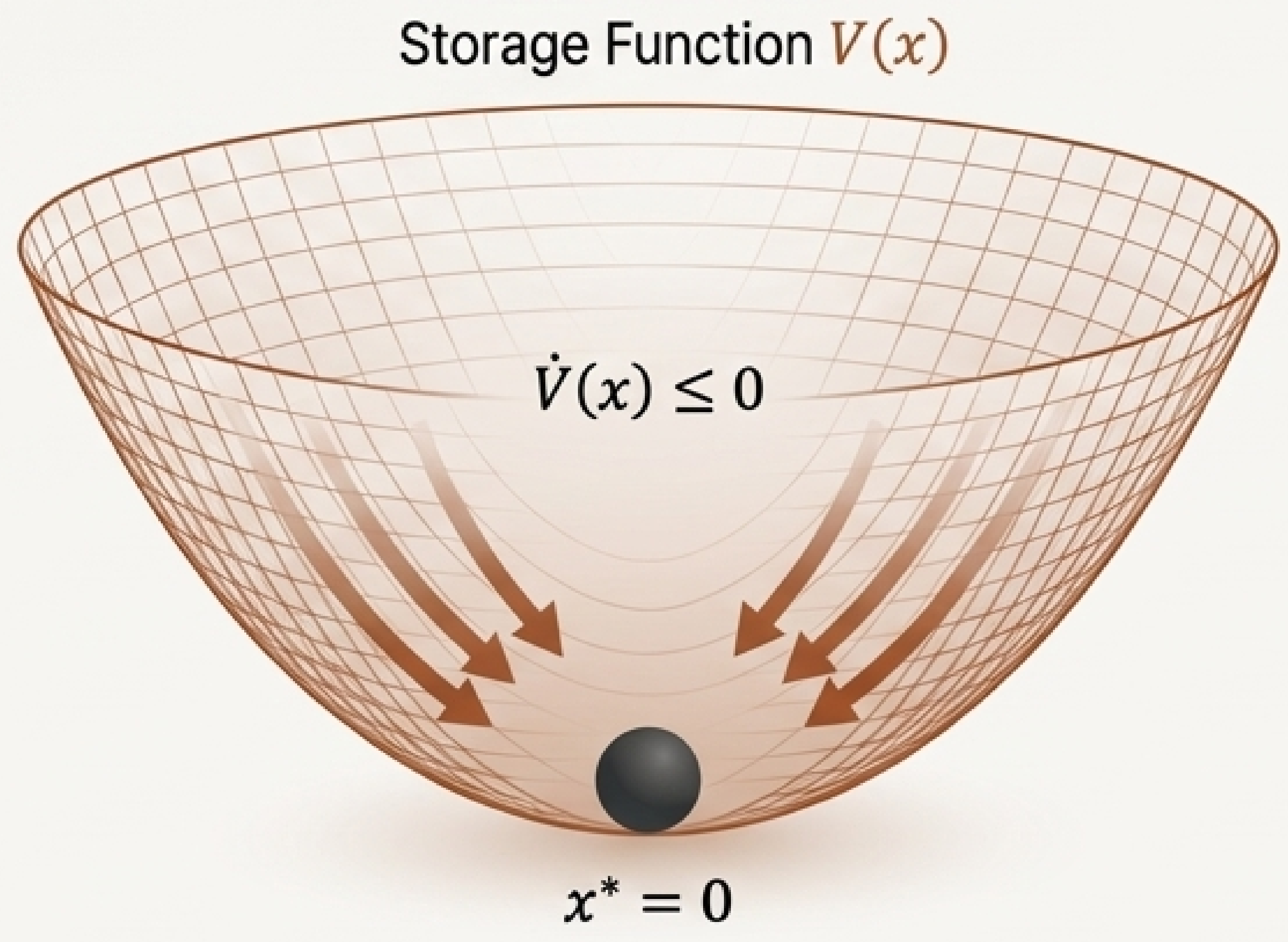
A fundamental link exists between passivity and the stability of an equilibrium point $x^* = 0$.

**Theorem:**

- An equilibrium point $x^* = 0$ of an unforced system ($u = 0$) is **Lyapunov stable** if:
  a) the system is passive, and
  b) the storage function $V(x)$ is continuously differentiable and positive definite (pdf).

**The Logic:**

1. The differential passivity inequality is: $\dot{V}(x) \leq y^T u$.
2. For the unforced system, we set the input $u = 0$.
3. The inequality simplifies to: $\dot{V}(x) \leq 0$.
4. A positive definite function $V(x)$ with a negative semidefinite derivative $\dot{V}(x)$ is the definition of a Lyapunov function that proves stability.



# Asymptotic stability requires strict energy dissipation and observability.

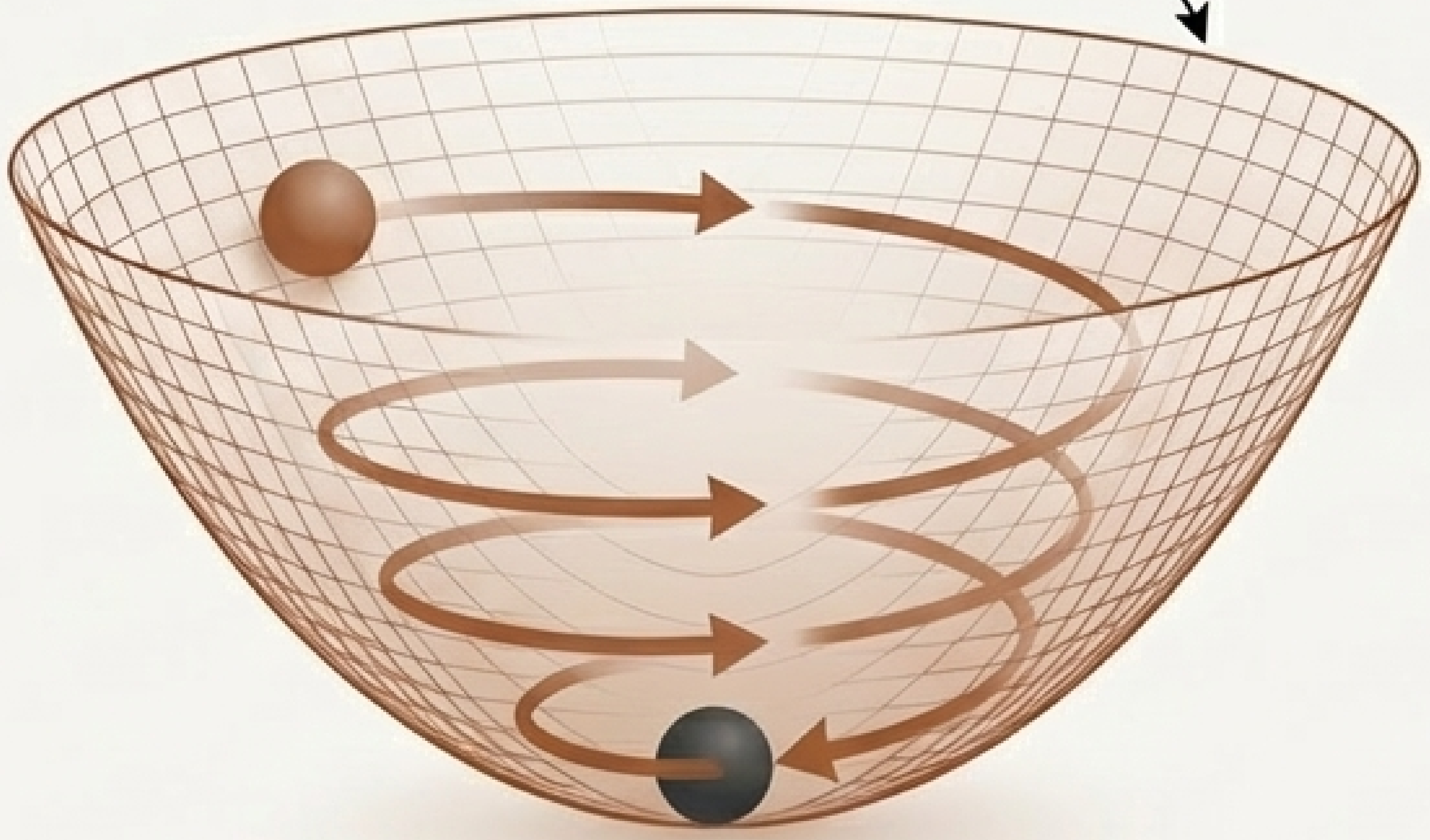
To ensure trajectories converge to the equilibrium $(x^* = 0)$, stronger conditions are needed. The equilibrium is asymptotically stable if any of the following hold:

a) The system is **state strictly passive**.
   - $\dot{V}(x(t)) \leq y^\top u - \Psi(x(t))$, where $\Psi(\cdot)$ is pdf.

b) The system is **output strictly passive** AND **zero-state observable**.
   - $\dot{V}(x(t)) \leq y^\top u - y^\top \rho(y)$, where $y^\top \rho(y)$ is pdf.

c) The system is **passive, zero-state observable,** $V(x)$ is pdf, and $\dot{V}(x) = 0$ only if $y = 0$.

**Global Stability:** If the storage function $V(x)$ is also **radially unbounded,** the stability conclusion is global.



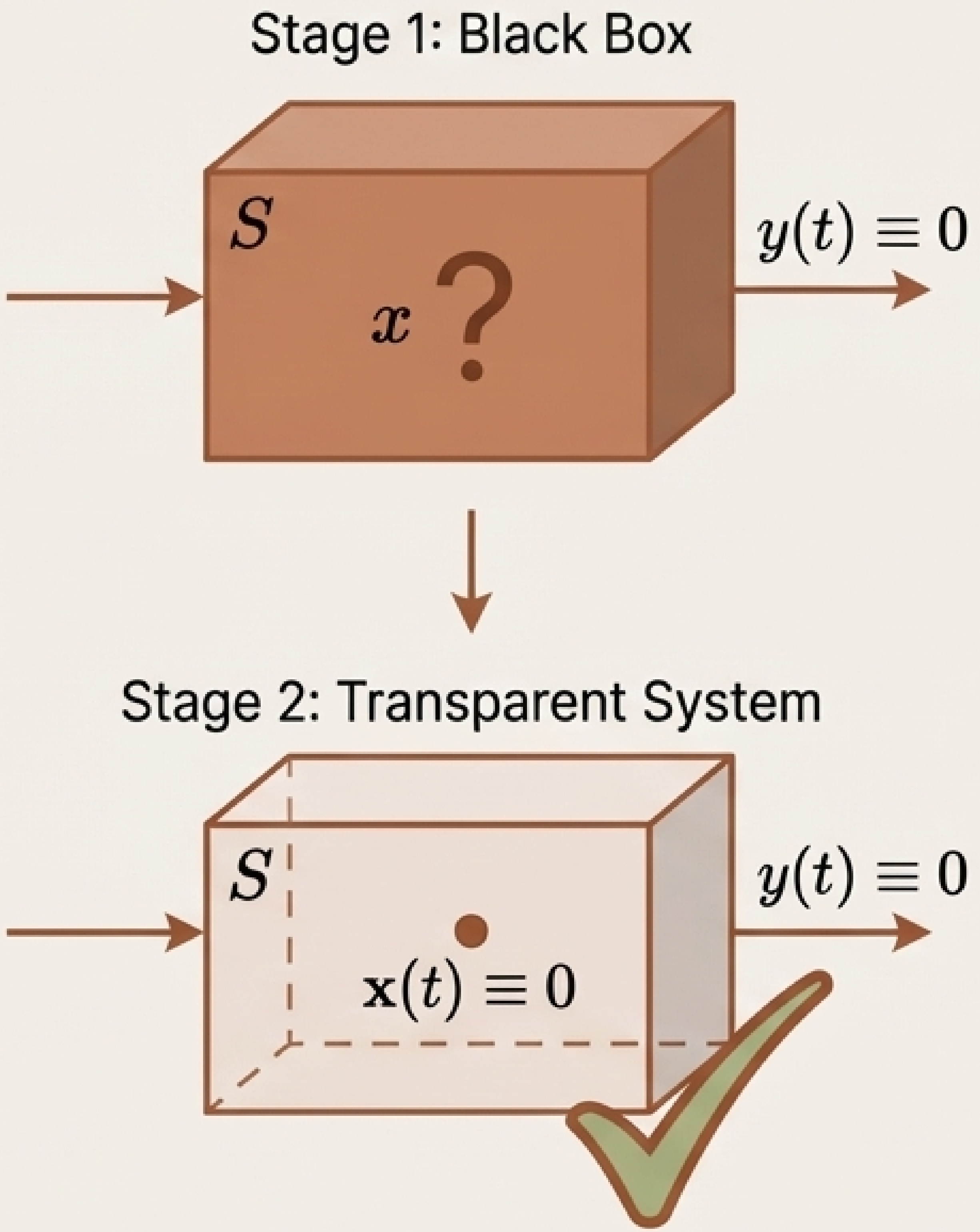
# Zero-State Observability: If the output is zero, must the state be zero?

This property is crucial for ensuring that a zero output implies the system is truly at its equilibrium state.

**Definition**: A system $\dot{x} = f(\mathbf{x}, u), y = h(\mathbf{x}, u)$ is zero-state observable if no solution of the unforced system $\dot{x} = f(\mathbf{x}, 0)$, besides the trivial solution $\mathbf{x}(t) \equiv 0$, can remain in the set $S = \{\mathbf{x} \mid h(\mathbf{x}, 0) = 0\}$.

**Verification Strategy**

1. Set input $u = 0$ in the system dynamics: $\dot{x} = f(\mathbf{x}, 0)$.
2. Find the set $S$ where the output is zero: $S = \{\mathbf{x} \mid h(\mathbf{x}, 0) = 0\}$.
3. Check if any non-zero trajectory $\mathbf{x}(t)$ can stay inside $S$ forever. If not, the system is zero-state observable.



# Passivity-Based Control (PBC) leverages these properties to design stabilizing controllers.

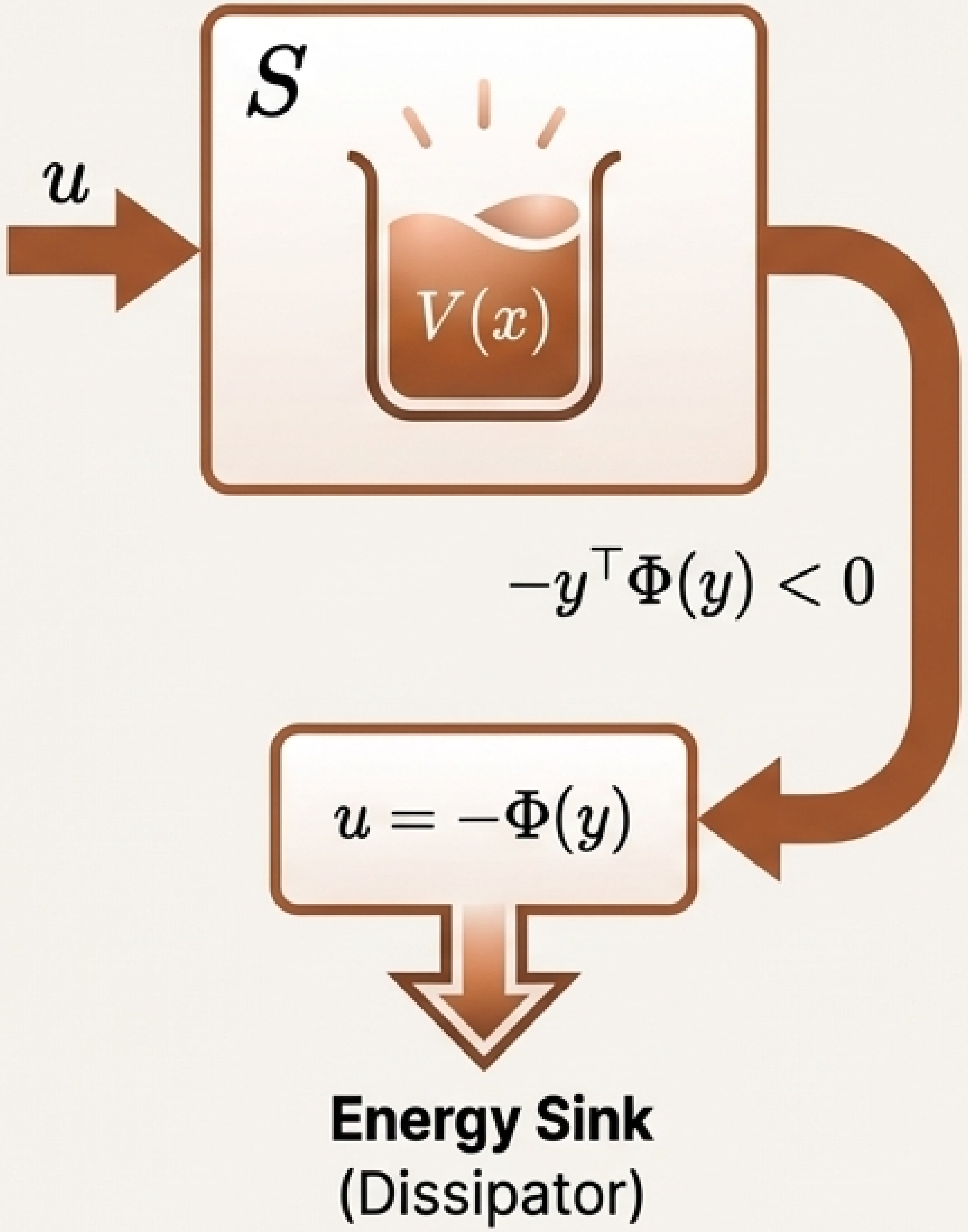
**Theorem of Passivity-Based Control:**
If a system is:

- **Passive** with a storage function $V(x)$ that is positive definite (pdf) and radially unbounded, AND
- **Zero-state observable**,

then the equilibrium $x^* = 0$ can be **globally asymptotically stabilized** with the feedback law:

$$u = -\Phi(y)$$

where $y^\top \Phi(y)$ is a positive definite function (e.g., $\Phi(y) = k * y$ for k>0).



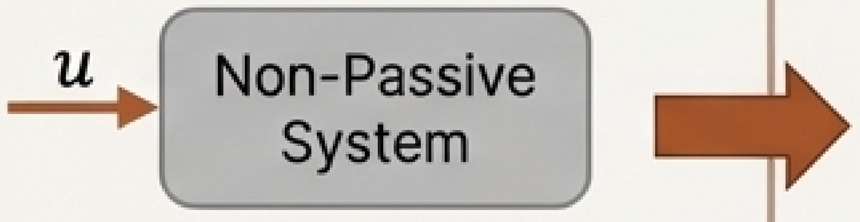

**Intuition:**

The controller $u = -\Phi(y)$ acts as an "**energy sink**". It ensures that the power supplied to the system, $y^\top u = -y^\top \Phi(y)$, is always negative when the output is non-zero.

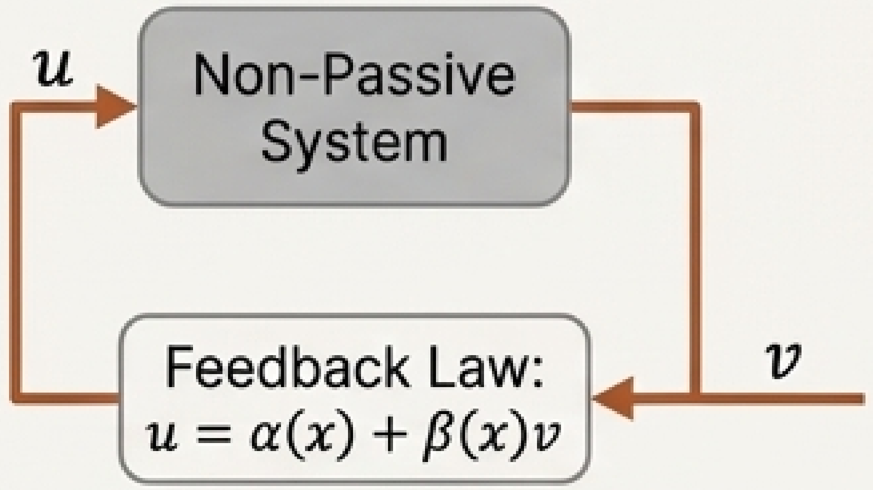
This actively removes energy from the system, forcing its state to the lowest energy level: the origin.

# Feedback Passivation can make a non-passive system passive.

If a system is not inherently passive, we can sometimes design a feedback law to render it passive. This is called **Feedback Passivation**. For a control-affine system $\dot{x} = f(x) + G(x)u$:

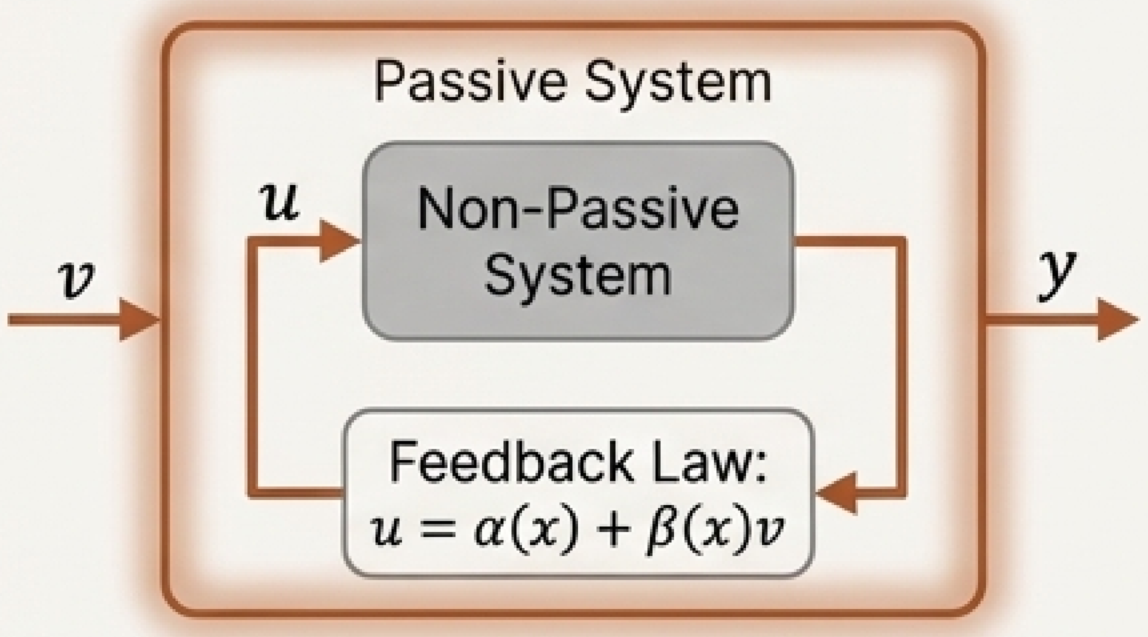



**Step 1:** Stabilize internal dynamics.




Find a feedback law $u = \alpha(x) + \beta(x)v$ and a storage function $V(x)$ such that the internal dynamics are stable. This requires making $\frac{\partial V}{\partial x}(f(x) + G(x)\alpha(x)) \leq 0$.

**Step 2:** Choose new output to achieve passivity.




A strategic choice is $y = \left[\frac{\partial V}{\partial x} G(x)\beta(x)\right]^{\top}$.

The transformed system with input $v$ and output $y$ is now **passive**. A stabilizing controller $v = -\Phi(y)$ can then be designed using the principle of **Passivity-Based Control**.

# Passivity: From Energy Concepts to Robust Control Design

## 1. A Formalization of Energy Dissipation.

Passivity is a mathematical framework for energy balance, defined by the inequality $\dot{V}(x) \leq y^\top u$. The storage function $V(x)$ $V(x)$ tracks the system's internal energy.

## 2. The Power of Compositionality.

Passivity is preserved under parallel and feedback interconnections. This is a rare and powerful property that enables the modular design of complex systems with guaranteed stability.

## 3. A Direct Path to Stability and Control.

Passivity directly implies Lyapunov stability. With stricter conditions like zero-state observability, it guarantees asymptotic stability and provides a constructive method (PBC) for designing globally stabilizing nonlinear controllers.