DYNAMICAL SYSTEMS: A Deep Dive into Feedback Linearization

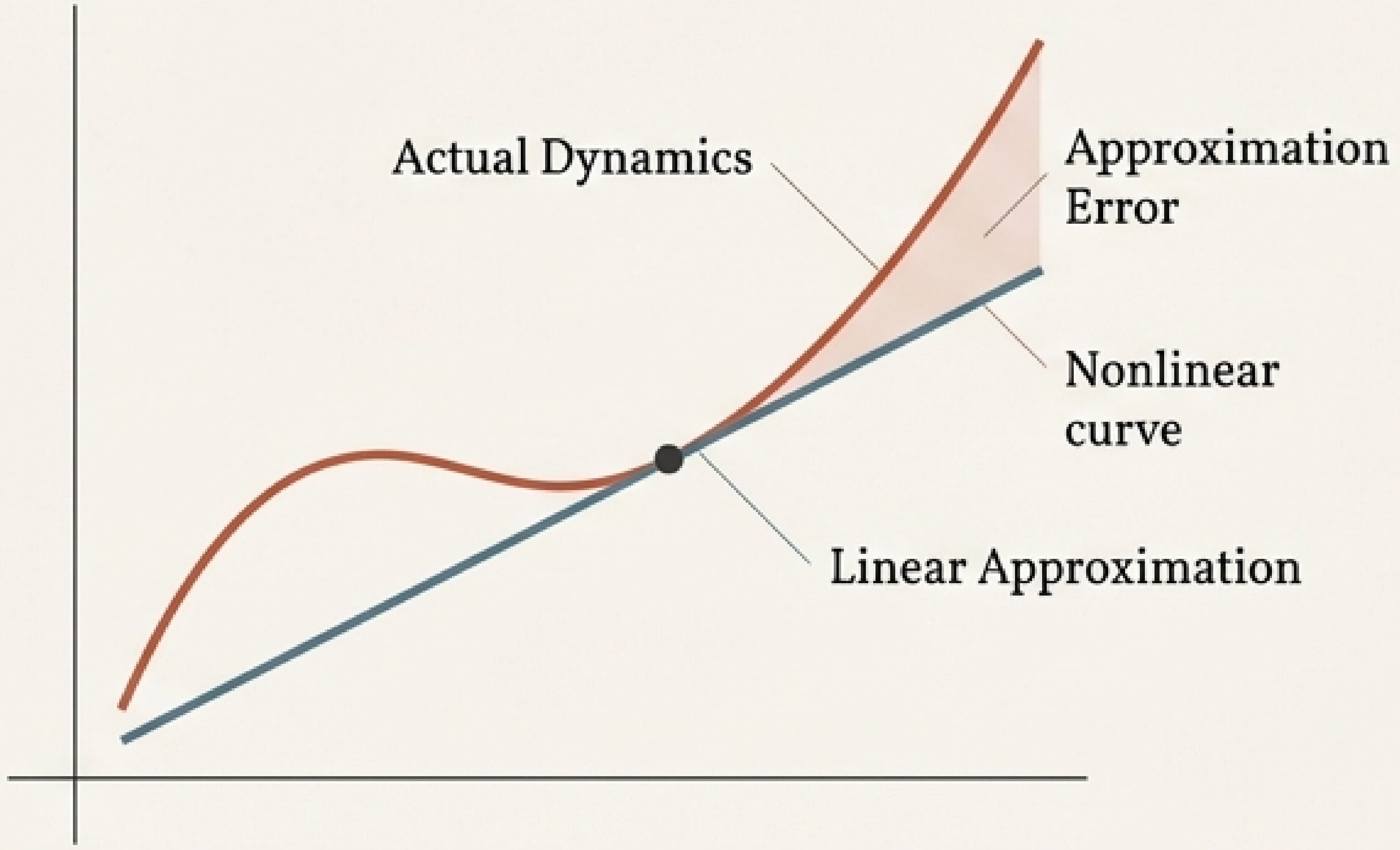
Transforming Nonlinear Complexity into Linear Simplicity

Based on Lecture Notes from the Chair of Automatic Control Engineering, Technical University of Munich.

Moving Beyond Approximation to Exact Transformation

Taylor Series Linearization

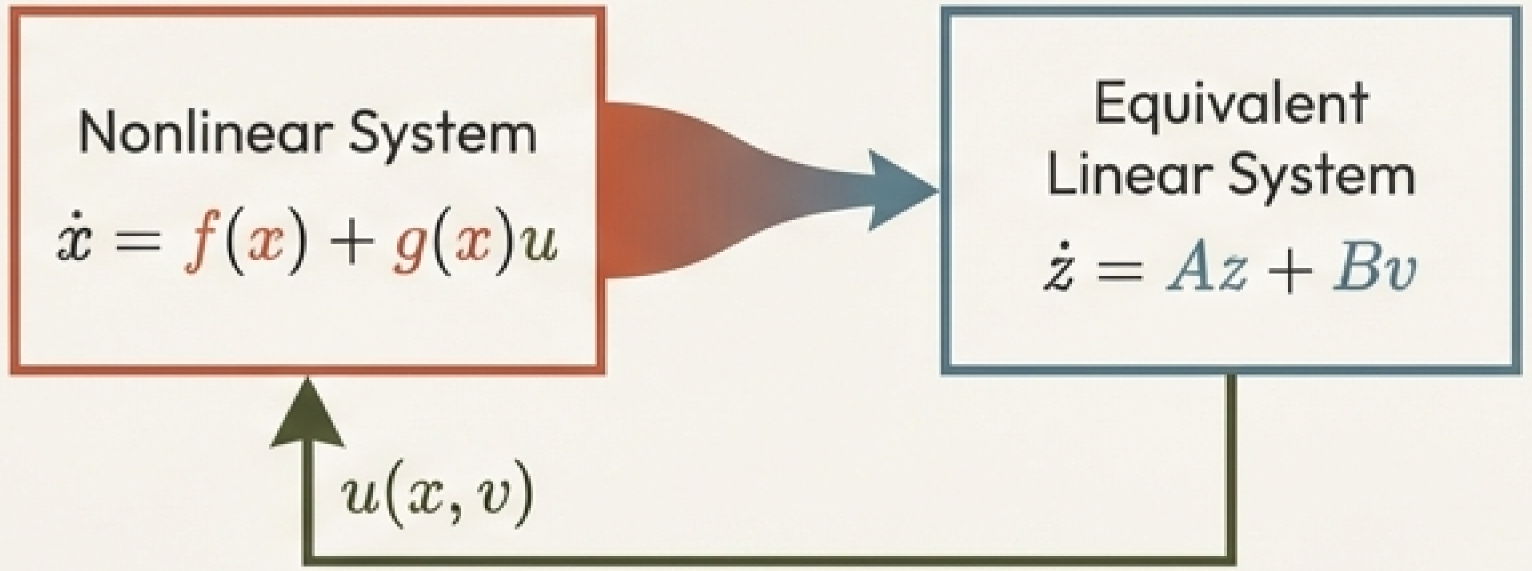
An approximation valid only in a small neighborhood around an operating point.



Based on Section 1.3, "Linearization… by Taylor Series Approximation".

Feedback Linearization

An *exact* system transformation, creating an equivalent linear system through a change of coordinates and feedback. Valid over a much larger region of the state space.



Based on Section 1.4, "Linearization… by State Feedback".

The Language of Nonlinear Systems

Lie Derivative

$$L_f h := \nabla h \cdot f$$

$$L_f^0 h = h, L_f^i h = L_f L_f^{i-1} h$$

Measures the rate of change of a function h along the vector field f.

Lie Bracket

$$\begin{aligned}[f, g] &:= L_f g - L_g f \\ &= \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g\end{aligned}$$

Captures the non-commutativity of flows along vector fields f and g.

Ad Operator

$$\mathrm{ad}_f^0 g = g,$$
$$\mathrm{ad}_f^i g = [f, \mathrm{ad}_f^{i-1} g]$$

Provides a compact notation for repeated Lie Brackets.

The Key Metric: Relative Degree (r)

How many times must we differentiate the output y before the input u appears?

Formal Definition

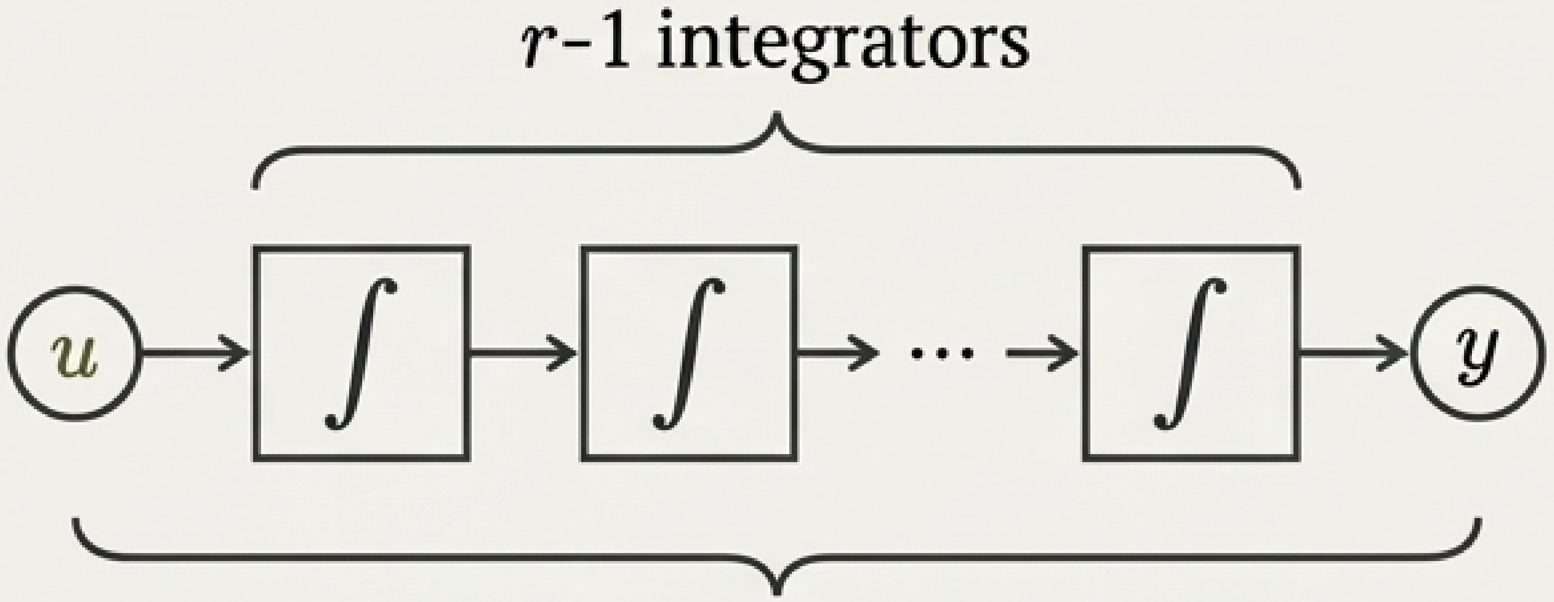
The system $\dot{x} = f(x) + g(x)u, y = h(x)$ has a relative degree r at point x° if:

1. $L_g L_f^k h(x) = 0$ for all x in the neighborhood of x° and for all $k < r - 1$.

   (The input u has no direct effect on $y, \dot{y}, \ldots,$ up to the (r-1)-th derivative.)

2. $L_g L_f^{r-1} h(x^\circ) \neq 0$

   (The input u finally appears explicitly in the expression for the r-th derivative of y.)



Relative Degree r = length of the shortest dynamic path

Method 1: Input/Output Linearization

Strategy: *Differentiate the output $y = h(x)$ exactly r times until the input u appears*

1. Differentiate

Compute derivatives of y until the r-th derivative reveals the input u:

$$y^{(r)} = \underline{L_f^r h(x)} + \underline{L_g L_f^{r-1} h(x) u}$$

2. Define Control Law Terms

Isolate the nonlinear parts of the equation:

$$a(x) := L_f^r h(x)$$

$$b(x) := L_g L_f^{r-1} h(x)$$

(Note: The definition of relative degree r guarantees that $b(x) \neq 0$ in the region of interest.)

3. Synthesize Feedback

Design the control input u to cancel the nonlinearities and introduce a new, simpler input v:

$$u(x, v) = \frac{1}{b(x)}(v - a(x))$$

Result

Substituting this u back into the equation for $y^{(r)}$ yields a perfectly linear input-output relationship:

$$y^{(r)} = v$$

The Result: A Perfect Integrator Chain

Under the feedback law $u(x, v)$, the system's behavior from the new input v to the output y y is now equivalent to a chain of r integrators in Controllable Canonical Form.

New State Definition

Define a new state vector z of dimension r:

$$z = [y, \dot{y}, \dots, y^{(r-1)}]^T$$

Linear Dynamics

$$\dot{z} = Az + Bv$$
$$y = Cz$$

Next Step

Standard linear control techniques (e.g., pole placement) can now be used to design a controller for v to place the poles of this new, simplified linear system.

Matrix A Companion Matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix B

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

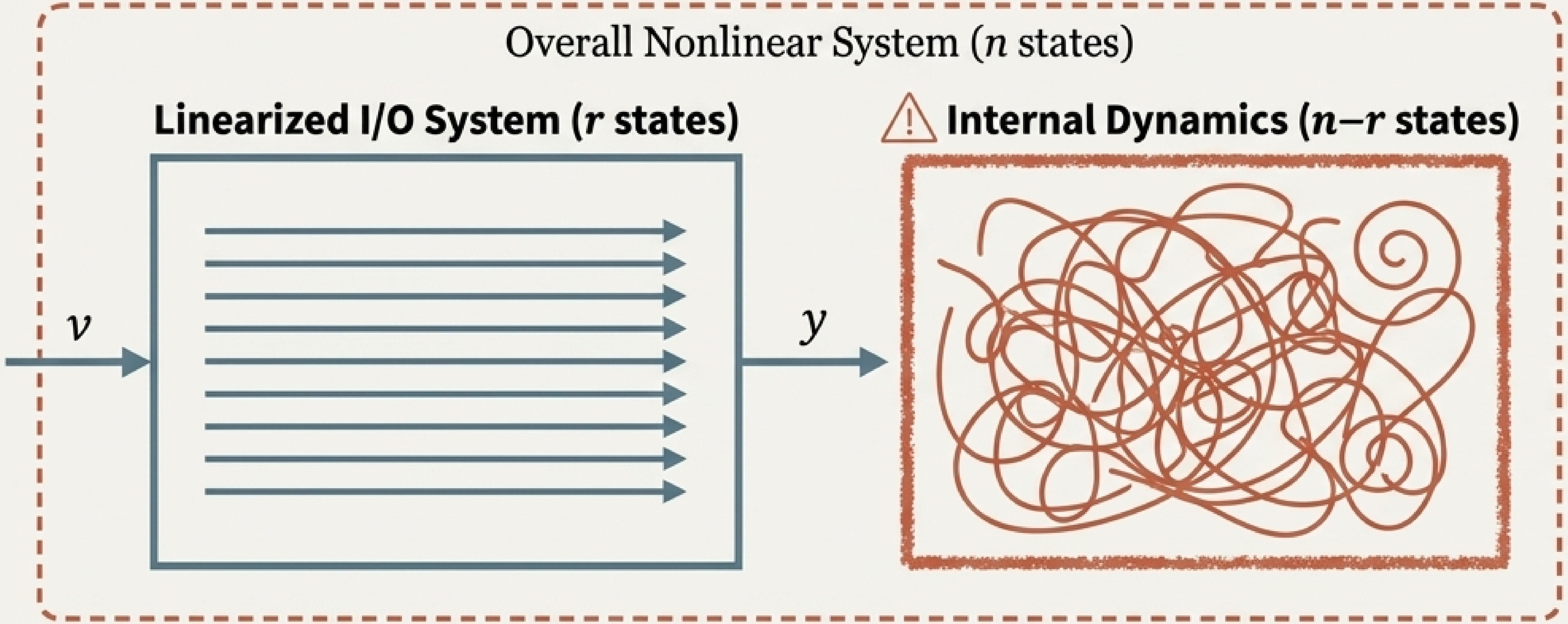
Matrix C

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Critical Caveat: Unseen Internal Dynamics

*What happens when the relative degree **r** is less than the system order **n**?*

I/O linearization only transforms r states. The remaining $n-r$ states form the **internal dynamics**, which are not controlled by v or observed through y.



If these internal dynamics are unstable, the entire system can become unstable even if the input-output behavior appears perfectly controlled.

This is the behavior of the internal dynamics when we force the output to be zero ($y(t) \equiv 0$). The stability of the zero dynamics is a crucial test for the viability of the I/O linearization controller.

Method 2: Full-State Linearization

Goal: Linearize the entire n-dimensional state dynamics, ensuring no internal dynamics by achieving $r = n$.

Strategy: Instead of using a given output $h(\mathbf{x})$, we must find a special scalar output function $y = \varphi_1(\mathbf{x})$ that is guaranteed to have a relative degree of n. This requires solving a specific system of Partial Differential Equations (PDEs).

$\mu^*(\mathbf{x})$ is a non-zero function of our choice.

$$\frac{\partial \varphi_1(\mathbf{x})}{\partial x} \cdot \begin{bmatrix} \mathbf{g}(\mathbf{x}) \\ \mathrm{ad}_f \mathbf{g}(\mathbf{x}) \\ \cdots \\ \mathrm{ad}_f^{(n-1)} \mathbf{g}(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} 0 \\ \cdots \\ 0 \\ \mu^*(\mathbf{x}) \end{bmatrix}$$

$S(\mathbf{x})$: The Reachability Matrix

The Result: A Completely Linear System

Process (Once $\varphi_1(x)$ is found)

Step 1: Construct the new n-dimensional state vector $\boldsymbol{z}$:

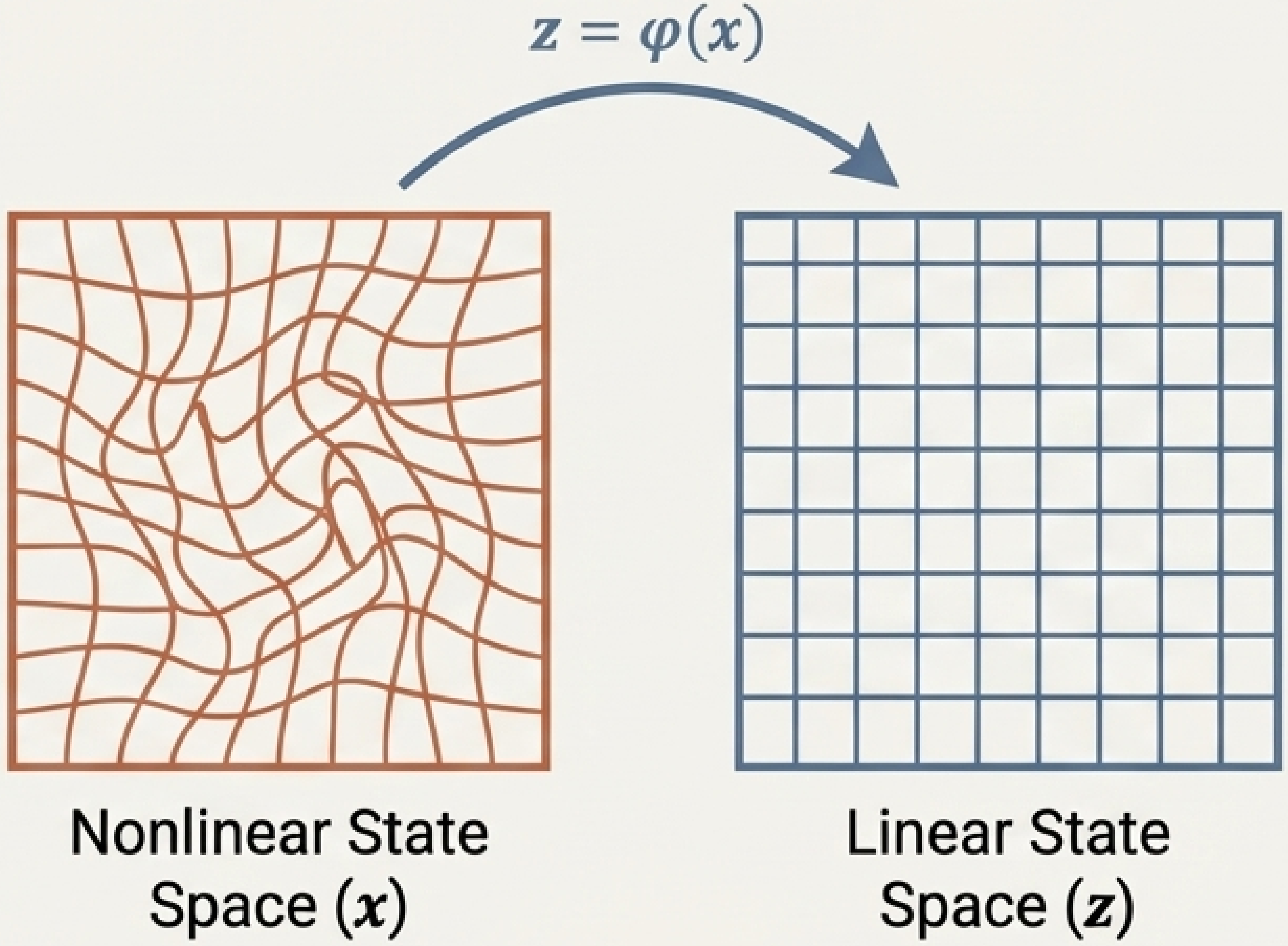
$$\boldsymbol{z} = \boldsymbol{\varphi}(\boldsymbol{x}) := \left[\varphi_1(x), L_f\varphi_1(x), \ldots, L_f^{n-1}\varphi_1(x)\right]^T$$

Step 2: Construct the control law $\boldsymbol{u}(x, v)$:

$$\boldsymbol{u}(x, v) = \frac{1}{L_g L_f^{n-1}\varphi_1(x)}\left(v - L_f^n\varphi_1(x)\right)$$

The Outcome

The transformed system $\boldsymbol{z}$ is now in a linear, controllable canonical form $\dot{\boldsymbol{z}} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{v}$, with **no internal dynamics**. The entire n-dimensional state is now controllable via the new input v.



NotebookLM

Condition 1 for Full-State Linearization: Reachability

The control input u must be able to influence the state in any direction in the n-dimensional state space.

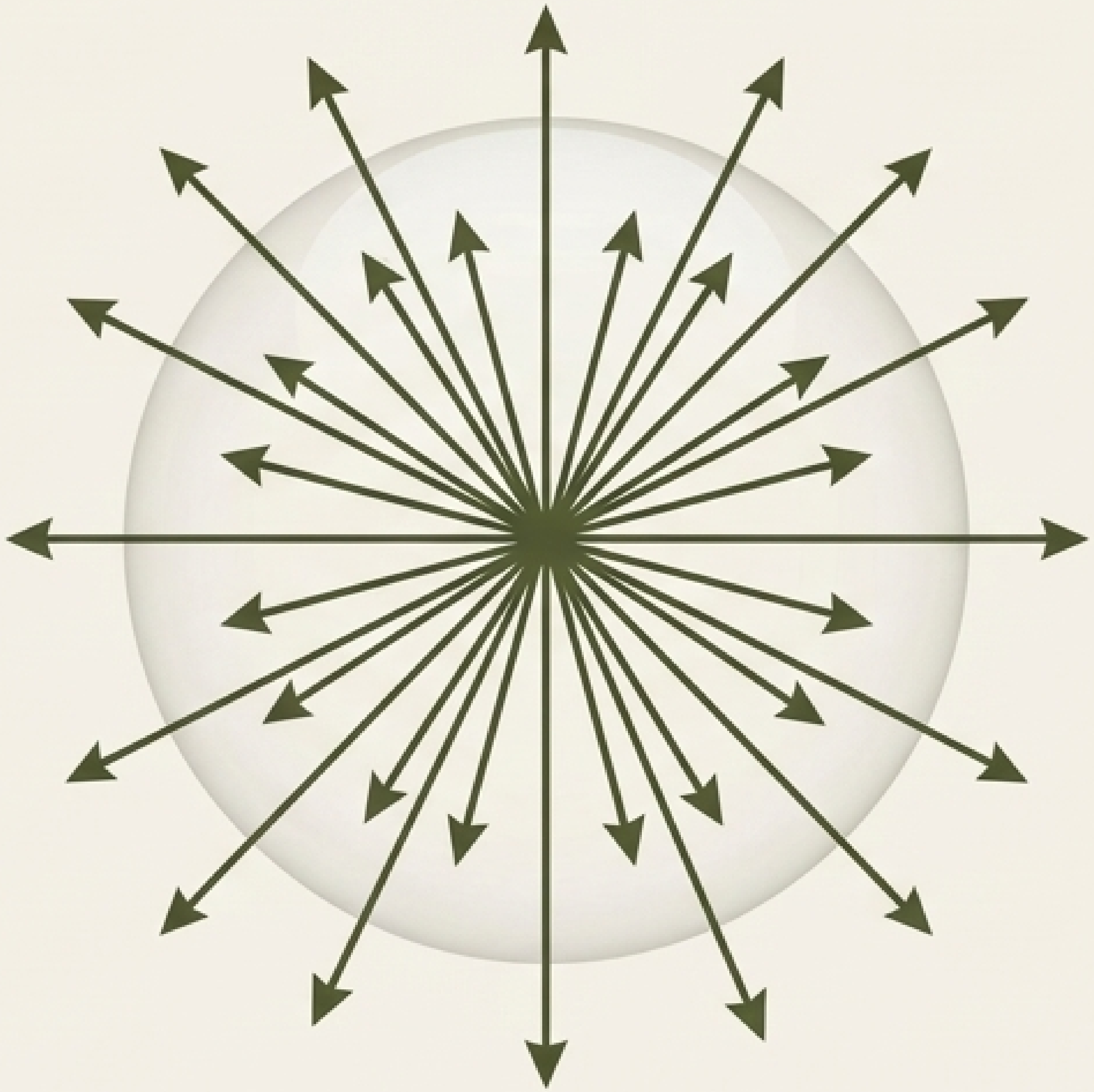
The Test

The system $\dot{x} = f(x) + g(x)u$ must be (locally) reachable. This is verified using the reachability matrix, $S(x)$:

$$S(x) = \left[g(x), \mathrm{ad}_f g(x), \ldots, \mathrm{ad}_f^{n-1} g(x)\right]$$

rank$(S(x)) = n$ for all x in the domain of interest.

Full Control Authority



Condition 2 for Full-State Linearization: Involutivity

The vector fields that define the directions unaffected by the input must form a “closed” set under the Lie bracket operation. This ensures the PDEs for $\varphi_1(x)$ are solvable.

The Test

The distribution (set of vector fields) $B(x)$ must be **involutive**.

$$B(x) = \{g(x), \mathrm{ad}_f\, g(x), \dots, \mathrm{ad}_f^{n-2}\, g(x)\}$$

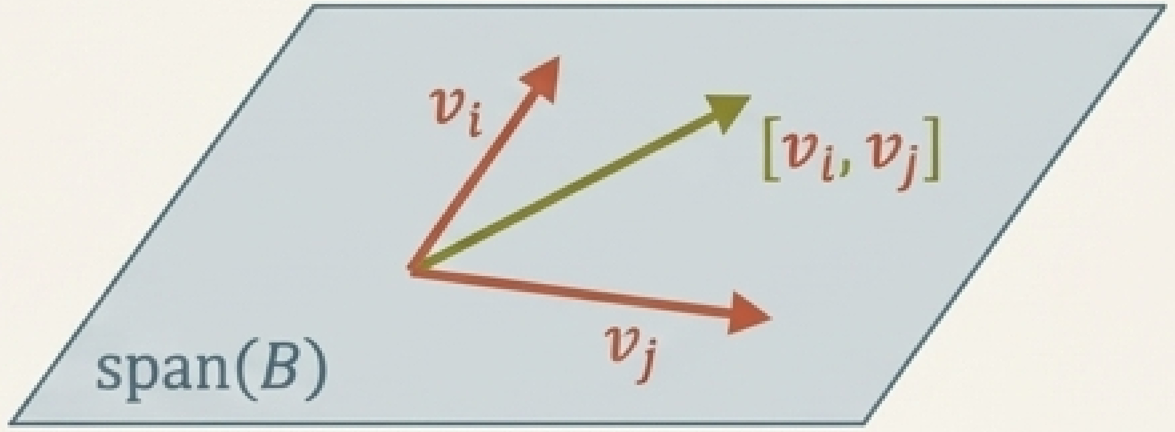
A set of vector fields is involutive if the Lie Bracket of any two vector fields in the set can be written as a linear combination of the vector fields already in the set:

$$[\mathrm{ad}_f^i\, g, \mathrm{ad}_f^j\, g] \in \mathrm{span}(B) \text{ for all } i, j \in \{0, \dots, n-2\}$$

A Sufficient Condition

While the formal test can be complex, a simpler way to confirm involutivity is to check if the vector fields in $B(x)$ are all constant vectors:

$$g, \mathrm{ad}_f\, g, \dots, \mathrm{ad}_f^{n-2}\, g \text{ are constant.}$$



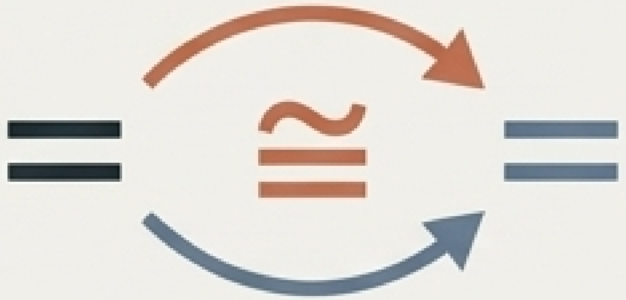
Choosing the Right Approach: I/O vs. Full-State

Input/Output Linearization	Full-State Linearization
• **Goal:** Linearize the relationship for a *given* output $y = h(x)$. • **Process:** Repeated differentiation of $h(x)$. • **Result:** A linear integrator chain of order r. • **Main Challenge:** Must separately analyze the stability of the $n-r$ **internal dynamics.** Potentially simpler but may be an incomplete solution.	• **Goal:** Linearize the *entire* state dynamics. • **Process:** Solve PDEs to find a *special* output $y = \varphi_1(x)$. • **Result:** A fully linear system of order n. No internal dynamics. • **Main Challenge:** Strict conditions must be met (Reachability & Involutivity). Mathematically more demanding but provides a complete solution.

Key Takeaways

An Exact Algebraic Transformation

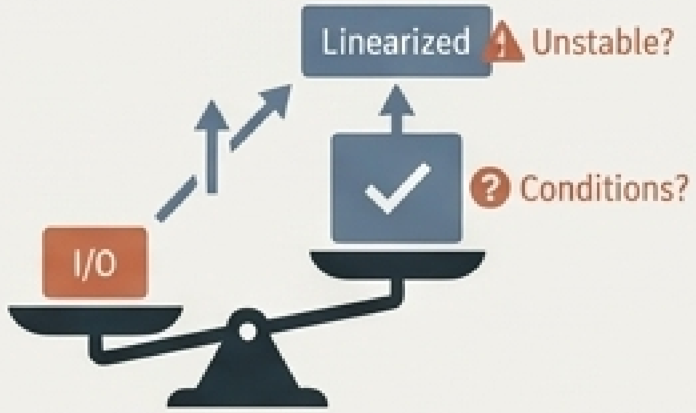
Feedback linearization is not an approximation like Taylor series. It algebraically transforms a nonlinear system into an equivalent linear one, valid over a large operating domain.



Two Methods, A Key Trade-Off

I/O Linearization is direct but risks instability if the hidden internal dynamics ($r < n$) are unstable.

Full-State Linearization is complete ($r = n$) but is only possible if strict ($r = n$) but is only possible if strict mathematical conditions (reachability, involutivity) are satisfied.



Geometred Niear Linearization

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The Power of Geometric Tools

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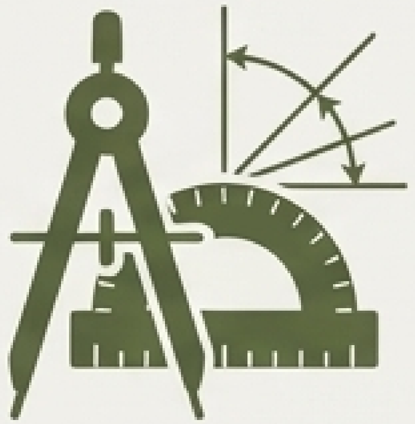

From Complexity to Simplicity